Key Ideas

- Let $T$ be a linear transformation and $S$ any set of points. Then $T(S)$ denotes the set of points you get by applying the transformation $T$ to each of the points of $S$ separately.

- For example, if $T$ is the linear transformation that rotates the plane counterclockwise by $90^\circ$ and $S$ is the set of points $(x, y)$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. We could apply the transformation separately to each point, and the collection of all points we get would be the set of points $T(S)$ which satisfy $-1 \leq x \leq 0$ and $0 \leq y \leq 1$.

- The first question we want to answer today is this: what does the determinant of the standard matrix for a transformation tell you about how it acts on shapes? For instance, the matrix of the transformation above (what is it again?) has determinant equal to 1. What is the geometric significance of that fact?

- **Theorem 10:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A$. If $S$ is any geometric object in the plane with finite area then

$$\{\text{Area of } T(S)\} = |\det A| \{\text{Area of } S\}.$$  

Similarly, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with standard matrix $A$. If $S$ is any geometric object in three dimensional space with finite volume then

$$\{\text{Volume of } T(S)\} = |\det A| \{\text{Volume of } S\}.$$  

- So, the geometric significance of the fact that the transformation in the last example had a determinant of 1 is that it was a rotation, and hence does not change the area of shapes that it acts on. Had the determinant had an absolute value of 2 we would have known that it was a transformation that doubles areas.

- If you took Math 221, do you remember using determinants there? They showed up in computing the **Jacobian**. It was essentially this geometric application that you were using there. Even though those changes of coordinates were not linear transformations, there is a linear transformation related to them much as the tangent line is associated to a differentiable function, and the key to translating an integral from one set of coordinates to the other is the determinant of that linear transformation, which records how the area or volume will change.

- **Notation:** Given an $n \times n$ matrix and a vector $x$ in $\mathbb{R}^n$ we write “$A_i(x)$” to denote the matrix that you get by replacing column $i$ in the matrix by the vector $x$. (Important: this is not matrix multiplication.)

- Suppose that $A$ is an invertible matrix and that $Ax = b$ for some vectors $x$ and $b$. We are going to be able to determine a remarkable formula relating the two vectors using determinants and the notation above. It will just take a few steps.
First, notice that
\[ AI_i(x) = A_i(b). \]
This is obvious because matrix multiplication can be seen as separately multiplying each column of the second matrix \((I_i(x))\) by the first matrix \((A)\). For any column but the \(i^{th}\) one, this yields the \(i^{th}\) column of \(A\). Since column \(i\) is the vector \(x\), we know that we will end up with \(b\) in that column! QED.

Now, take the determinants of each side. Notice that \(\det(AI_i(x)) = \det(A) \det(I_i(x))\) and \(\det(I_i(x)) = x_i\) (the component of the vector \(x\) in position \(i\)).

Putting this together we get \textbf{Theorem 7}:

\[ x_i = \frac{\det A_i(b)}{\det A}. \]

This is known as \textit{Cramer’s Rule}. Note that it allows us to find a single component of the solution to \(Ax = b\) without finding the inverse or the entire solution.

A further use of Cramer’s rule is an explicit way to write \(A^{-1}\). Note that the \(j^{th}\) column of \(A^{-1}\) is the solution to \(A \cdot x = e_j\). Then we can use Cramer’s rule to write a formula for \(A^{-1}\):

\[ \{(i, j)\text{-entry of } A^{-1}\} = \frac{\det A_i(e_j)}{\det A}. \]

Note that we can make this even simpler, because expanding the determinant of \(A_i(e_j)\) down the \(i^{th}\) column we see that this is the same as what we previously called “cofactors” \(C_{ji}\) (see handout from 2/14/11).

This leads to the formula

\[ A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det A} \text{adj} A \]

Note that the \textit{adjugate matrix} \(\text{adj } A\) is the transpose of what you might expect it to be. (The entry in the \(i^{th}\) row and \(j^{th}\) column is the cofactor for the \(j^{th}\) row and \(i^{th}\) column of \(A\)!

There is an important distinction between the \textit{algorithm} we learned in Section 2.2 for finding \(A^{-1}\) and this formula we just found. First, I would like to emphasize that the algorithm from 2.2 is probably \textit{easier!} So, the importance of this new formula is not ease of use. The difference here is that this formula is \textit{explicit}. The algorithm involves steps, but there is no telling which steps will be needed. On the other hand, regardless of which invertible matrix \(A\) you have, the formula above does give the inverse.

Another significance of the inverse formula from Cramer’s rule is that you can compute just a single entry of the matrix (if you happen to only need to know the top left corner, for example).
• Finally, note the importance of \( \det A \) in the formula above...as in the \( 2 \times 2 \) case it shows up in the denominator. (See exercise 18.)
• “So why are we learning Cramer’s rule?” A small child learning addition and multiplication has little appreciation for why these things might be useful later in their life. Similarly, it is difficult for students to get much appreciation for the algebraic structure of linear algebra at the time that they are learning it. Perhaps the most important lesson of Cramer’s Rule at this point is this: There is a beautiful a deep structure underlying linear algebra which is not obvious when one just thinks in terms of linear systems of equations. However, Cramer’s rule plays a more useful role in higher mathematics. For instance, in Math 323 it is used to find solutions to differential equations and in Math 245 it is used to determine how much of a problem will be caused by “round off errors”.

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**Homework**

**Section 3.3:** Read the section and do these problems: 1–6*, 11–18, 27, 28, 31, 32

**Extra Problem***: “Let \( C \) be the circle of radius one centered at the origin and \( T \) be the linear transformation with standard matrix

\[
\begin{bmatrix}
2 & 3 \\
3 & 2
\end{bmatrix}.
\]

What is the area of the ellipse \( T(C) \)?”