Key Ideas: Properties of Determinants

- Let us start with an example which will give us a chance to practice the “expansion” method for computing determinants from last time.

**Question 1:** Compute $\begin{vmatrix} 1 & 9 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{vmatrix}$.

- **Theorem 3** tells us how the elementary row operations affect a determinant:
  a) Adding a multiple of one row to another **does not change** the determinant.
  b) Exchanging rows **changes the sign**.
  c) Multiplying an entire row by a scalar multiplies the determinant by that same scalar.

- (As I mentioned last time, in most books I’ve read, the definition of determinant is that $\det(A)$ is the unique function from $n \times n$ matrices to $\mathbb{R}$ such that $\det(I) = 1$ and the above theorem is true.)

- **Important Observation:** We can use just (a) and (b) from above to put any matrix $A$ into echelon form (not necessarily reduced echelon form which would require using (c)). This gives us:

  **Another Way to Compute Determinants:** Put the matrix $A$ into echelon form $U$ using the row operations (a) and (b) only. Keep track of how many times you use (b) and call this number $r$. This matrix is then an upper-triangular matrix and so by Theorem 2 from last time we have that $\det(U)$ is just the product of the diagonal elements and so

  $$\det(A) = (-1)^r \left( \text{product of diagonal elements of } U \right).$$

- Note how the previous formula works out differently in the invertible and non-invertible cases:
  - If $A$ is invertible, then all of these diagonal elements are non-zero and so $\det(A)$ is non-zero.
  - If $A$ is not invertible then some of the diagonal elements are zero and so $\det(A) = 0$.

- The previous observation is summarized in **Theorem 4**: $\det(A) \neq 0$ is equivalent to $A$ being invertible. (Corollary: $\det(A) = 0$ is equivalent to saying that the columns of $A$ are linearly dependent. This is how determinants are used in differential equations, as a test for linear independence!)

- An important property is that $\det(AB) = \det(A) \det(B)$ (**Theorem 6**). From this we can easily check that $\det(A^{-1}) = 1 / \det(A)$. Note that it is **not** generally true that $\det(A + B) = \det(A) + \det(B)$!
• It is not hard to check that 
  \[ \det A = \det A^\top. \]

  (It follows from the equivalence of the formulas for the determinant as an expansion down a column or across a row.) This is not only important for the purposes of computing determinants. It has a useful corollary: You can perform column operations on a matrix just like the row operations and they will have the same effect on the determinants. (Also, note that \( \det A = 0 \) is equivalent to the the rows of \( A \) being linearly dependent.)

**Homework**

**Section 3.2:** Read the section and then do these problems. 1–9, 10⋆, 15, 16⋆, 17–23, 29, 32