• **Lemma 1:** If \( AB = I_m \) then every row of \( A \) has a pivot position.

We can prove this using a combination of our usual “proof technique” and one additional fact.

- **Step One:** Let \( b \) be any vector in \( \mathbb{R}^m \). I claim that the vector \( w = Bb \) solves the equation \( Ax = b \).
  In particular we can check that \( Aw = A(Bb) = ABb = Ib = b \).

- **Step Two:** We know from Theorem 4 in Chapter 1 that \( Ax = b \) has a solution for every \( b \) in \( \mathbb{R}^m \) if and only if the matrix \( A \) has a pivot in every row.
  Combining these two facts, we see that \( AB = I_m \) implies that \( A \) has a pivot in every row!

• **Lemma 2:** If \( BA = I \) then every column of \( A \) is a pivot column.

We can prove this one as follows: Suppose \( x \) is a vector that satisfies the homogeneous equation \( Ax = 0 \). We can multiply both sides of that equation by the matrix \( B \). On the left side we get \( BAx = B0 = 0 \) and on the right we get \( B0 = 0 \). This means that, if \( BA = 0 \), then the only solution to \( Ax = 0 \) is the trivial solution. We already know that this implies also that every column of \( A \) has a pivot. (If any column was pivot-less then there would be non-trivial solutions!)

• So, although I asked you to trust me and believe that this was a characterization of invertibility, we can now say for certain:

**Theorem:** A matrix is invertible *if and only if* it is square and row equivalent to the identity matrix. (We already know that if a matrix is row equivalent to the identity that we can find the inverse by the procedure we learned last time. On the other hand, it follows from the two lemmas above that if a matrix is invertible, it must be square and row equivalent to the identity – these are the identity matrices are the only ones in RREF for which every row and every column has a pivot.)

• So, if you have a matrix that is *not* square there is no question of it being invertible. However, if it *is* square, how can you tell? There are lots of *equivalent* ways to determine if a square matrix is invertible, and these are listed for you in **Theorem 8** on page 129.
**Theorem 8 The Invertible Matrix Theorem (IMT):** For an $n \times n$ matrix $A$, the following are either all true or all false:

a. $A$ is invertible.
b. $A$ is row equivalent to the identity matrix.
c. $A$ has $n$ pivot positions.
d. The equation $Ax = 0$ has only the trivial solution.
e. The columns of $A$ are linearly independent.
f. The linear transformation $T(x) = Ax$ is one-to-one.
g. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$.
h. The columns of $A$ span $\mathbb{R}^n$.
i. The linear transformation $T(x) = Ax$ is onto.
j. There is an $n \times n$ matrix $C$ such that $CA = I$.
k. There is an $n \times n$ matrix $D$ such that $AD = I$.
l. $A^\top$ is invertible.

- It is true that you will have to commit all of the things listed in Theorem 8 to memory at some point, but it is not really as hard as it may seem. If you think about it, many of these things are consequences of other things that you already know. (For example, you already know that an $n \times n$ matrix is row equivalent to the identity matrix if and only if every column is a pivot column, which is equivalent to saying that the columns are linearly independent!)

- It is useful to keep in mind the following fact: For $n \times n$ matrices, the question of whether every row has a pivot and the question of whether every column has a pivot are the same. That is, for a *square* matrix, the only way for every row to contain a pivot is for every column to contain a pivot, and *vice versa*.

- In terms of linear transformations, it both makes intuitive sense and follows from previous results (summarized in Theorem 9) that $T$ is invertible if and only if its standard matrix is invertible. (That is, there is a linear transformation $S$ so that $S(T(v)) = v$ and $T(S(v)) = v$.)

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**Homework**

**Section 2.3:** Read the section and then do these problems. 1–8, (do 9 and 10 if you have software or a calculator that can compute RREF), 11, 12*, 15–19, 33, 34, 36