Key Ideas

- **Fast Matrix Multiplication:** According to the definition, this matrix product is
  \[
  \begin{pmatrix}
  1 & 2 & 3 \\
  -1 & 2 & 3 \\
  5 & 0 & 5
  \end{pmatrix}
  \begin{pmatrix}
  2 \\
  1 \\
  -1
  \end{pmatrix}
  = 2 \begin{pmatrix}
  1 \\
  -1
  \end{pmatrix}
  + 1 \begin{pmatrix}
  2 \\
  0
  \end{pmatrix}
  - 1 \begin{pmatrix}
  3 \\
  5
  \end{pmatrix}
  = \begin{pmatrix}
  1 \\
  -3
  \end{pmatrix}.
  \]
  
  Note that the first entry in the final result is made just from the first row of the matrix, the second entry from just the second row of the matrix, and so on. This is the basis for an easy and quick way to multiply without having to write so much down. I will start class by showing this on the board.

- **Group Project Proof:** Most of you did pretty well on the proof on the group project, but I had to give lots of hints. Let me briefly review what we did and how the proof should look.

  First, think about what the claim said. It said that if \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are any two solutions to the equation \( A\mathbf{x} = \mathbf{b} \) and \( m \) is any number then \( \mathbf{w} = m\mathbf{v}_1 + (1 - m)\mathbf{v}_2 \) is also a solution to that same equation. It is pretty obvious that this is true if \( m = 1 \) or \( m = 0 \), but that it is true for every value for \( m \) (giving us infinitely many solutions) is pretty amazing. So amazing, you might doubt it.

  That’s what a proof is supposed to be. It is supposed to be a demonstration of a fact so convincing that it is no longer possible to doubt it. As you write it, you should imagine a very skeptical person saying “Yes, but how do you know you can do that?” You should justify each step so that even that skeptic won’t be able to deny the validity of the claim after reading your proof.

  This is what my proof of the claim would look like:
  
  \[
  M\mathbf{w} = M (m\mathbf{v}_1 + (1 - m)\mathbf{v}_2) \quad \text{(that’s just the definition of \( \mathbf{w} \))}
  = M (m\mathbf{v}_1) + M ((1 - m)\mathbf{v}_2) \quad \text{(distribution of matrix multiplication)}
  = m(M\mathbf{v}_1) + (1 - m)(M\mathbf{v}_2) \quad \text{(matrix mult commutes with scalar mult)}
  = m\mathbf{b} + (1 - m)\mathbf{b} \quad \text{(since \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) solve \( A\mathbf{x} = \mathbf{b} \))}
  = m\mathbf{b} + \mathbf{b} - m\mathbf{b} \quad \text{(distribute scalar mult)}
  = \mathbf{b} \quad \text{(cancel \( m\mathbf{b} \) with \(-m\mathbf{b}\))}
  \]

- **Theorem 4:** I somehow failed to mention this important result from Section 1.4 (which would have been useful in answering the homework question that I was asked at the board). It says that the following are equivalent conditions on an \( m \times n \) matrix:
  1. Every vector in \( \mathbb{R}^m \) can be written as a linear combination of the columns of \( A \).
  2. There is a solution to the equation \( A\mathbf{x} = \mathbf{b} \) for every vector \( \mathbf{b} \) in \( \mathbb{R}^m \).
  3. \( A \) has a pivot position in every row.
**Question 1:** If \( A \) is a \( 3 \times 2 \) matrix, must there be a \( 3 \)-vector \( b \) such that \( Ax = b \) has no solution? (Yes, because \( A \) can have at most two pivots, so it cannot have a pivot in every row. Then, by Theorem 4, it is not true that \( Ax = b \) has a solution for every \( b \).)

- **Flashback: Algebra:** Why is it that the solution to \( (x - 3)(x - 4) = 0 \) is \( x = 3 \) or \( x = 4 \)?. When multiplying numbers (scalars), the equation \( ab = 0 \) is true if one of the numbers \( a \) or \( b \) is zero. Moreover, it is only true when \( a \) or \( b \) is zero! This is not true for matrix multiplication.

- **Flashback: Calculus:** Consider the equation \( \frac{d}{dx}f = 3x^2 - 5 \). What is the general solution to this equation? It would be \( f(x) = x^3 - 5x + C \), right? Let’s think of that as having two different pieces. One piece is the \( x^3 - 5x \), which is a solution to the equation. The other part, the \( C \), does not solve this equation, but it does solve the equation \( \frac{d}{dx}f = 0 \) (replace the righthand side with \( 0 \)). We will see that the same sort of decomposition applies for the matrix equation \( Ax = b \). (Note: This is not a coincidence. As we will see toward the end of the course, we can really view \( \frac{d}{dx} \) as being a matrix and polynomials as being vectors and understand the “\( + C \)” as being a consequence of linear algebra!)

- It is useful to write the solution set to \( Ax = b \) in vector parametric form, as linear combination of constant vectors with one coefficient equal to the number one and the rest being the free variables. The procedure for finding such solutions in practice (described on page 53) is simple enough: Find the solution as we have developed earlier by row reducing a matrix. Collect the coefficients of the free variable \( x_i \) as a vector and call it \( v_i \). Collect the constant terms (independent of the free variables) in the solution together in the form of a vector and call it \( p \). Then the solution is

\[
x = p + x_{i_1}v_{i_1} + x_{i_2}v_{i_2} + \cdots + x_{i_m}v_{i_m}
\]

where \( x_{i_1}, \ldots, x_{i_m} \) are the free variables.

**Question 2:** Derive the general solution to the equation

\[
\begin{pmatrix}
1 & 2 & 0 & 3 & 1 \\
0 & 0 & 1 & 1 & 1 \\
1 & 2 & 0 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
=
\begin{pmatrix}
2 \\
1 \\
2
\end{pmatrix}
\]

in vector parametric form. Hint: If I did it right, the answer turns out to be

\[
x =
\begin{pmatrix}
2 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
+ x_2
\begin{pmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}
+ x_5
\begin{pmatrix}
-1 \\
0 \\
-1 \\
0 \\
1
\end{pmatrix}
\]

**Goal:** We want to understand the structure of that solution.

- **Trivial Solution to the Homogeneous Equation:** The homogeneous equation \( Ax = 0 \) always has at least one solution: the zero vector \( x = 0 \). We call that “the trivial solution”.

• **Non-Trivial Solutions to the Homogeneous Equation:** The interesting question is, are there any *other* solutions to \( Ax = 0 \)? If it has *any* other solutions, then it has infinitely many, which only happens if there are “free variables”. So, we can conclude: **The homogeneous equation \( Ax = 0 \) has a nontrivial solution if and only if the matrix \( A \) has a non-pivot column.** (Note that we are stating this in terms of the pivot columns of \( A \) itself and not the augmented matrix, which always has a non-pivot column in the last column!)

• Recall that \( \text{Span}\{v_1, \ldots, v_n\} \) is the set of vectors that you can get as a linear combination of the vectors \( v_i \). If there is just one vector \( (n = 1) \) then this is a line, if there are two \( (n = 2) \) then it is (probably) a plane, and so on for higher dimensional “spaces”. However, no matter what the \( v_i \)'s are or how big \( n \) is, these spans always include the origin \( 0 \) (the vector with all zero entries). Why?

• If we add the vector \( p \) to every vector in the span of some others then we are *translating*, for example, pushing it away from the origin while keeping its shape.

• We can get a complete understanding of the geometric structure of the solution space of the vector equation \( Ax = b \) as the span of some set of vectors translated by one vector. This is **Theorem 6**:

   Suppose the equation \( Ax = b \) is consistent for some given \( b \) and let \( p \) be a solution. Then the solution set of \( Ax = b \) is the set of all vectors of the form \( p + v_h \) where \( v_h \) is any solution of the homogenous equation \( Ax = 0 \).

• You will *prove* this theorem as part of your homework (problem 25). Hint: it follows from the distributive property of matrix multiplication.

---

**Homework**

**Section 1.5:** Read the section and then do these problems. 1–12, 15, 16*, 25, 29, 30*, 31–35, 36