Math 203: Handout 01/10/11

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Why not to take Linear Algebra

- Do not make the mistake of taking this class because you expect it to be easy. Both the name (isn’t “algebra” a class in high school and aren’t “lines” the simple geometric objects?) and the number (Math 203) are misleading. This class is hard! You can expect to learn not only some new techniques, but also lots of new ideas in this class. You will need to read the textbook and do lots of homework.
- Also, do not expect this class to be about multiplying matrices and adding vectors. Matrices and vectors are part of the language we will be using, it is no more accurate to say that they are the subject of the class than to say that War and Peace is about words. The class will be about vector spaces and linear transformations.

Why you should take Linear Algebra

- Well, as I said above, the class will be hard, but as I learned too late myself, it is the hard classes that you get the most out of. You might even appreciate the difficulty for its own sake; think of it as brain exercise. However, there are many reasons to learn linear algebra other than for a mental workout.
- The abstract concept of “vector spaces” and “linear transformations” are key to most of the subjects you think of being mathematical, and many that you do not. In fact, if you are going to go into any area of science, engineering or computer science that depends heavily on mathematics, you will most certainly encounter problems that would be difficult or impossible to solve without linear algebra. (Even topics that you might not think of as especially mathematical, such as computer graphics, involve a great deal of linear algebra.)
- Linear algebra is an important step on the way to more advanced topics in mathematics. The ideas we develop in this class will be built upon and utilized in classes such as abstract algebra, differential equations, numerical methods and analysis. Most importantly, this will be a good opportunity to begin learning how you can prove theorems rigorously. “Proof” is not just a meaningless exercise in mathematics, it is the way we see our abstract mathematical universe. For many of you, this class will be the first course in which you are required to contribute independent proofs.
- Some of the ideas in this class are just cool: Does the Pythagorean theorem work in 928,735-dimensional space? (It does, and that it good because it was needed to solve the very real problem described on page 373!) What would algebra be like if there was no commutative law; that is, what if \( a \times b \neq b \times a \)? (Linear algebra is one example of such an algebra, so we’ll get to see...and you need this kind of weird algebra for everything from solving Rubic’s Cube to explaining the structure of the hydrogen atom.) How can you choose a set of points in the plane that look like some rigid object rotated in three-dimensional space? (See the floating “L” on the class syllabus!)

Unfortunately, I can’t jump right in to talking about these things in detail on the first day. We will begin the class with some really simple stuff that will not seem very unusual to you: solving linear equations. In fact, through the first chapter of the book we will mostly only be learning terminology and notation, but no really new ideas. But don’t let this fool you. After that we’ll be taking some ‘quantum leaps’ into unknown territory. First, let’s get talking about linear equations.
A linear equation is an equation of the form
\[ a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \]
where \( a_i \) and \( b \) are some fixed constants. It is called linear because we do not see any products or powers of the variables \( x_1, \ldots, x_n \). We generally are interested in finding solutions to the equation, which would be sets of numbers \( (x_1, \ldots, x_n) \) that satisfy the equation. A system of linear equations is a set of equations of this type which must be solved simultaneously. One key motivating idea in this class will be an attempt to solve and to understand the solution sets of systems of linear equations.

First Surprise: A system of linear equations either has one solution, no solution or infinitely many solutions. It never has exactly 3 solutions or 908 solutions. By tomorrow we will be able to prove this rigorously, and later in the course we will even understand it geometrically, but for now I merely ask you to believe me and to consider the case of two-variables as a good example.

The linear system
\[ x_1 + x_2 = 2 \quad \text{and} \quad 3x_1 + cx_2 = d \]
has only the solution \( x_1 = 2, x_2 = 0 \) if \( c \neq 3 \) and \( d = 6 \). (Think about them graphically to understand how I know.) On the other hand, if \( c = 3 \) and \( d = 9 \) then there are no common solutions. (Again, think graphically.) Finally, if \( c = 3 \) and \( d = 6 \) then there are infinitely many solutions. (Why?)

If the system has no solutions we say it is inconsistent. This means either that one of the equations is impossible (like \( 0 = 3 \)) or that some of the equations “contradict” each other. In other words, the system is consistent if the solution sets to each of the equations intersect somewhere. (Next time we will learn a formal procedure for finding out if a system is consistent...but for now you’ve just got to think it through.)

Note that we are not really very interested in the equations themselves, but only in their solutions. We want to know the set of values that we could give the variables \( x_i \) so that the equations are all true, this is called the solution set. Different systems can have the same solution sets. (For example, The system \( x_1 + x_2 + x_3 = 2 \) and \( x_1 - x_2 = 0 \) has the same solution set as \( 2x_1 + x_3 = 2 \) and \( -x_1 + x_2 = 0 \).) If that is the case, we say those systems are equivalent.

It will be convenient for now, and very important later, to represent systems of linear equations by a matrix (an array of numbers written as group of rows of numbers each having the same length). The matrix of just the coefficients of the variables is called the coefficient matrix and one that also includes a final column representing the values on the right side of the equations is called the augmented matrix. (See page 4-5.)

The three basic row operations are: exchanging rows, scaling rows and adding rows. The important fact about these is that doing any of these operations to an augmented matrix for a linear system takes you to the augmented matrix of an equivalent system. So, you can use them to simplify the system to the point that you can solve it (by eliminating extra terms to get a “triangular” matrix) and it can also show you when a system is inconsistent. (We will see this in greater detail in our next class.)

Homework

Read Section 1.1 and do problems: 1–17, 18*, 19–21, 22*