KEY IDEAS FROM SECTION 5.5: THE METHOD OF SUBSTITUTION

• Remember that in Section 3.11, we learned that the symbols “$dx$”, “$dy$” and “$dt$” that we use so often in class are called **differentials**. You may also have noticed that one of these differentials is always part of any definite integral:

\[
\int_{a}^{b} f(x) \, dx.
\]

Until now, that $dx$ just seemed to sit there doing nothing. In fact, it is an important part of the integral. Today we will learn a method that makes use of it. First, however, we must learn how to work with differentials.

• **The Algebra of Differentials**: Given any equation of the form $y = f(x)$ (where a function of the variable $x$ is given a name such as $y$), one can also write a corresponding equation for differentials

\[
dy = f'(x) \, dx.
\]

Moreover, you can do ordinary algebra with this new equation. For example, notice that it is true (dividing both sides by $dx$) that $dy/dx = f'(x)$! Here is a more involved example. Suppose $u = 3x^2 - 17$. Then it is true that $du = 6x \, dx$, but also that $\frac{1}{6} \, du = x \, dx$.

• **The Method of Substitution**: Often a problem of antidifferentiation can be made simpler by making **substitutions** of the differentials. For example, do you know how to antidifferentiate $x \sin(x^2)$? This one seems hard...we certainly don’t have a function like this in any of our tables of elementary antiderivatives. However, if you consider $u = x^2$, then $du = 2x \, dx$ and $\frac{1}{2} \, du = x \, dx$. So

\[
\int x \sin(x^2) \, dx = \int \sin(x^2) \, (x \, dx) = \int \sin(u) \left( \frac{1}{2} \, du \right) = \frac{1}{2} \int \sin(u) \, du.
\]

But we know $\frac{1}{2} \int \sin(u) \, du = -\frac{1}{2} \cos(u) + C$. In fact, resubstituting back for $u$, we get $-\frac{1}{2} \cos(x^2) + C$, which is the general antiderivative we were looking for!

• **A Good Idea**: You can never be certain whether substitution will help you with a particular problem until you try. However, it is usually a good idea to try the substitution $u = f(x)$ for an antiderivative if you see $f(x)$ **inside** another function and you see some constant multiple of its derivative ($cf'(x)$) as a common factor of the whole integrand.

• It is pretty simple now for us to determine an antiderivative for the tangent function. Using the method of substitution we can figure out that

\[
\int \tan(x) \, dx = \ln|\sec(x)| + C.
\]
Sometimes, the method of substitution can be tricky. Try these examples
\[ \int \frac{\ln(x)}{x} \, dx \quad \int x^5 \sqrt{1 + x^2} \, dx. \]
In the first one, it may be difficult to figure out what you want to use for \( u \), but if you let \( u = \ln(x) \), it works out pretty easily. In the second, things work out if you let \( u = 1 + x^2 \), but in a not very obvious way! (You need to make use of the fact that \( x^2 = u - 1 \).

**Definite Integrals with Substitution:** One of our main interests in antidifferentiation is its use in evaluating definite integrals. (Using the second part of the FTC.) There are two different ways in which you can use substitution for a definite integral:

- Suppose we want to evaluate \( \int_{-1}^{1} x^2 \sqrt{1 - x^3} \, dx \).

One thing that you can do, of course, is to first find the antiderivative using substitution (in this case \( u = 1 - x^3 \)):
\[ \int x^2 \sqrt{1 - x^3} \, dx = -\frac{1}{3} \int \sqrt{u} \, du = -\frac{2}{9} u^{3/2} + C = -\frac{2}{9} (1 - x^3)^{3/2} + C \]

Then use this to evaluate the definite integral
\[ \int_{-1}^{1} x^2 \sqrt{1 - x^3} \, dx = -\frac{2}{9} (1 - x^3)^{3/2} \bigg|_{-1}^{1} = \frac{4\sqrt{2}}{9}. \]

- Or, we could use another method for the same integral. Let \( u = 1 - x^3 \) again, but now change the endpoints of the integral as well. If \( u = g(x) \), then the endpoints become \( g(a) \) and \( g(b) \) instead of \( a \) and \( b \). Here, instead of \( x = -1 \) for the lower endpoint, we have \( u = 1 - (-1)^3 = 2 \) and instead of \( x = 1 \) of the upper endpoint we have \( u = 1 - 1^3 = 0 \):
\[ -\frac{1}{3} \int_{2}^{0} \sqrt{u} \, du = -\frac{2}{9} u^{3/2} \bigg|_{2}^{0} = \frac{4\sqrt{2}}{9}. \]

Of course, we get the same thing both ways. If you want to see the second method work really well, try
\[ \int_{-1}^{1} x \sec(1 - x^2) \, dx. \]

**Homework**

**Web Problems:** The 16 problems in Web Assign are due Monday and are intended to prepare you for the questions that will appear on the final exam.

**Note:** On Friday we will do more examples of substitution, but we will also start to prepare for the final exam. I will teach you a few new things – things that will be on the final but which I did not cover carefully enough the first time we discussed them. I will also distribute a “review” sheet that will help you prepare for the final.