Key Ideas: from Section 5.4

An Interpretation of the FTC: Net Change:

- When I introduced the Riemann sums \( R_n, M_n, \) and \( L_n \) they seemed to be coming out of nowhere. However, I can show you some examples in which you would use them naturally. Suppose, for example, that I measured the velocity of a bicycle at 2 minute intervals and found this:

<table>
<thead>
<tr>
<th>time (minutes)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity (feet/minute)</td>
<td>10</td>
<td>45</td>
<td>55</td>
<td>47</td>
</tr>
</tbody>
</table>

How would we estimate how far the bicycle travelled during that time? You would end up using a Riemann sum because multiplying a speed (such as 45 ft/min) by the amount of time (2 minutes) would give you a good idea of how far it travelled during those two minutes. This is true in general, the Riemann sum would be a natural way to estimate the change in a function \( F(x) \) given values of its rate of change \( F'(x) \). Since the Riemann sum is also a good estimate for the definite integral, it is not very surprising that we learn today how the definite integral turns an instantaneous rate of change into the net amount that the original quantity changes.

- Recall that if a function \( F(x) \) measures some real quantity (like a property of an object in the real world) then its derivative \( F'(x) \) measures the instantaneous rate of change of that quantity with respect to the variable. (See the handouts from September 13th for more information.)

- The Fundamental Theorem of Calculus says something useful (and pretty simple) in the case of such a rate of change. Suppose that \( F(x) \) is some changing quantity and that \( f(x) = F'(x) \) is its rate of change. Then the FTC says:

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

In words, this says that by integrating the rate of change from \( x = a \) to \( x = b \), you get the amount that \( F \) changes between the start and end of that interval.

- As usual, the easiest example to consider is position and velocity. Let \( v(t) \) be the instantaneous velocity of a moving object at time \( t \). In this case, the “total change theorem” says: if you integrate \( v(t) \) from \( t = a \) to \( t = b \) you will get the distance between the position of the object at those times. Or, in geometrical terms, the areas on the graph of velocity represent distances (in opposite directions depending on whether it is above or below the \( x \)-axis)!
**Question 1:** If an object is travelling at velocity \( v(t) = 24t + 3t^2 \) feet/second at time \( t \) seconds, how far does it travel between time \( t = 2 \) and time \( t = 3 \)?

Answer:

\[
\int_{2}^{3} (24t + 3t^2) \, dt = 12t^2 + t^3 \bigg|_{2}^{3} = 79 \text{ feet}.
\]

**Question 2:** An object is travelling at velocity \( v(t) = 100 \sin(\pi t) \) meters/hour at time \( t \) hours. Look at the graph to determine the value of \( \int_{0}^{2} v(t) \, dt \) without doing any computations and interpret.

Answer: Clearly the integral is zero. This does not mean that the object did not move at all. It only means that at time \( t = 2 \) hours it is back at the same position it started in at time \( t = 0 \) hours.

- The same idea (integrating the rate of change to get the net change) applies in every situation: rate of water flowing out of a tank, the interest rate in your bank account, the rate at which a drug is being eliminated from a patient’s bloodstream, etc. See examples on pages 394–396.

**Question 3:** The figure below shows the rate of water flowing in (and out) of a bathtub measured in gallons/minute. The tub was first filled, and then after a brief pause, was drained.

1. Use the graph to evaluate the integral:

\[
\int_{0}^{5} f(t) \, dt = ?
\]

2. What is the physical meaning of your answer to part (a)?

3. Evaluate \( \int_{0}^{11} f(t) \, dt \) and explain why your answer is what we would expect.

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**Homework**

**Web Problems:** These problems are due just before class on the Tuesday after break. That’s a long time, but don’t wait until the last minute.

**Upcoming Tests:** Don’t forget that you have a test coming up in this class on December 2nd. Also, be aware that our final exam will be on 17 December at 8AM in 219 Maybank.