Key Ideas from Section 4.4: L’Hospital’s Rule

- Suppose we know the limit of \( f(x) \) and the limit of \( g(x) \) as \( x \) goes to \( a \)... what can we say about the limit of \( \frac{f(x)}{g(x)} \)?

Suppose \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} g(x) = L_2 \). Then:

- If \( L_1 \) is some real number and \( L_2 \) is a non-zero real number, then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2} \).

- If \( L_1 = 0 \) and \( L_2 \) is not, then \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \).

- If \( L_2 = 0 \) and \( L_1 \) is not then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist. (If \( a \) is a real number, in this case we say that \( x = a \) is a vertical asymptote.)

- If \( L_2 = \pm \infty \) and \( L_1 \) is not then \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \).

- If \( L_1 = \pm \infty \) and \( L_2 \) is a real number then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist (and \( x = a \) is a vertical asymptote if \( a \) is finite).

- If \( L_1 \) and \( L_2 \) are both zero or both infinite, then this is the most interesting case, because we cannot say whether \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists and what it is equal to if it does. These are called “indeterminate forms” for the limit.

- To use this you may need to know some things like:

\[
\begin{align*}
\lim_{x \to \infty} x^n &= \infty \quad (n > 0) \\
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to -\infty} e^x &= 0 \\
\lim_{x \to \infty} x^n &= 0 \quad (n < 0) \\
\lim_{x \to \infty} \ln(x) &= \infty \\
\lim_{x \to 0^+} \ln(x) &= -\infty.
\end{align*}
\]

- **Caveat:** It is very important to realize that “0/0” and “\( \infty/\infty \)” are not the values of the limits, but just a description of their form. In fact, a limit in these indeterminate forms can have any value or not exist. For instance, note that

\[
\lim_{x \to 1} \frac{x^2 + cx - 2x - c + 1}{x - 1} = c.
\]
• There are sometimes ways to use old-fashioned algebra to evaluate these indeterminate forms. For instance, the limit above can be found by factoring \((x - 1)\) out of the numerator and cancelling (as we would have done on the first test). The trick we learned in class last time (dividing the numerator and denominator by the thing that grows the fastest) is another way to handle indeterminate forms when \(a = \infty\).

**Question 1:** Let \(f(x) = x^2 - 1\) and \(g(x) = x^3 + 2\). What are the limits of \(f\) and \(g\) as \(x\) goes to \(\infty\)? What is the limit of their ratio, \(f/g\)?

• However, simple algebra may not work in all cases!

**Question 2:** Let \(f(x) = \ln(x)\) and \(g(x) = e^{x-1} - 1\). What are the limits of \(f\) and \(g\) as \(x\) goes to \(1\)? What is the limit of their ratio, \(f/g\)?

➤ One useful “trick” for evaluating limits is “L'Hospital’s rule” (Section 4.4). This says that if you are evaluating the limit \(\lim_{x \to a} \frac{f(x)}{g(x)}\) (where here \(a\) can also be \(\pm \infty\)) and you know that both \(f\) and \(g\) go to infinity or both go to zero, then this limit is the same as \(\lim_{x \to a} \frac{f'(x)}{g'(x)}\).

➤ Warning: This can only be applied to an indeterminate form. If you try to apply this rule to a limit of a fraction in any of the other forms, then you will end up with the wrong value.

• Sometimes you need to apply the technique more than once. Be sure to simplify between applications! And be sure to check that you still have an indeterminate form before trying again.

**Question 3:** Try this technique on the previous examples and also on \(\lim_{x \to 0} \frac{\sin(x)}{e^x - 1}\) and \(\lim_{x \to \infty} \frac{e^x}{x^2}\).

• There are also ways to use this for other indeterminate forms (e.g. those involving sums or powers). The trick in those cases is to somehow convert them to the indeterminate form above. For instance if \(\lim f(x) = 0\) and \(\lim g(x) = \infty\) then the product \(f(x)g(x)\) does not have an obvious limit. So instead consider \(\lim f(x)g(x) = \lim f(x)/g(x)\) which is in the “zero over zero” form. Also, if \(f(x)\) and \(g(x)\) both have infinite limits, then \(f(x) - g(x)\) is an indeterminate form. A trick that sometimes works there is to get a common denominator and write it as a single fraction. Finally, if taking the limit of \(f(x)g(x)\) produces an indeterminate form of \(0^0\), \(\infty^0\) or \(1^\infty\) then it is useful to write \(f(x)g(x) = e^{g(x)\ln(f(x))}\) and take the limit of \(g(x)\ln(f(x))\) as an indeterminate product.

**Question 4:** Try these:

\[
\lim_{x \to 0^+} x \ln(x), \quad \lim_{x \to \pi/2^-} \sec(x) - \tan(x), \quad \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}.
\]

(If I don’t get a chance to do these in class, they are all in the book on pages 202 and 203.)

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**Homework**

**Web Problems:** Set “HW 10/20” due on Friday.