Key Ideas from Section 3.4: The Chain Rule

- We know a shortcut for finding the derivative of $e^x$, $x^2$, and $\sin(x)$. We know a shortcut for their products, quotients and sums. However, there is another way to put these functions together: composition. This means putting one function inside the other function. (More specifically, it means making a function by putting your input into one function, then putting the output of that function into another function as input.) For instance:

  $$e^{\sin(x)}, \sin(x^2), \sin(e^{x^2}).$$

(We sometimes write $f \circ g(x) = f(g(x))$ and say “$f$ composed with $g$.”)

- A “Real” Example: Suppose $P(x)$ gives the air pressure (in atmospheres) at a height of $x$ feet above sea level. Suppose also that $h(t)$ gives the height of an airplane (in feet above sea level) at $t$ minutes after take-off. What is the meaning of $P(h(12)) = .8$? This is a practical composition of functions since it tells us something relevant. So, let us call the composition $R(t) = P \circ h(t)$, which gives the air pressure around the plane at time $t$. The question we will address today is: what is a formula for $R'(t)$ in terms of the functions $P$, $h$ and their derivatives? I think we can see from this example that it will involve a product of the derivatives! This may seem surprising, but it makes sense. Suppose $h(15) = 10,000$ (the plane is at 10,000 feet fifteen minutes into the flight), $h'(15) = 20$ (the plane is rising at a rate of 20 feet per minute) and $P'(10,000) = -.001$ (if you are at 10,000 feet and you go up a bit more, the pressure decreases by .001 atmosphere for each foot of the additional rise). The sign of $R'(15)$ should tell us whether the air pressure around the plane is increasing or decreasing at 15 minutes, and the value should tell us how quickly. But, it makes sense to me that this will be $P'(10,000) \times h'(15) = -.02$ atmospheres per minute because the first number says the plane is going up 20 feet per minute and the second says that the pressure decreases by .001 atmosphere for each foot! This suggests that the derivative of a composition is a product of derivatives.

- The derivative of a composition $P(x) = f(g(x))$ is given by the “shortcut”:

  $$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

  or

  $$\frac{dP}{dx} = \frac{dP}{dg} \cdot \frac{dg}{dx}$$

This is the chain rule. In class, I will explain the chain rule using the airplane/pressure example, but probably won’t prove it. The proof of the chain rule is almost as easy as the second equation in the box above makes it look. (See page 202 for the rigorous proof.)
We saw earlier that the derivative of $f(x) \times g(x)$ is not the product of $f'(x)$ and $g'(x)$ as you might expect. Now we see that the derivative of a composition is given by the product of the derivatives of $f$ and $g$—but it is important to note that it is $f'(g(x))$ and not $f'(x)$ that appears.

- Perhaps more useful than the formulas in the box above is the idea of a composition of an “outside function” and an “inside function”, and thinking of the chain rule as a way of peeling off the outside layer. (The derivative of such a composition takes the derivative of the outside function, leaving the inside function intact, and then multiplies by the derivative of the inside function.)

- This “peeling away” approach is especially useful when one must differentiate a composition of more than two functions. Then, one must peel away several layers, one at a time, like getting to the center of an onion! At each step you only need to differentiate the outermost “layer” of onion, so long as you can multiply by the derivative of what is inside it (which may or may not require more “peeling”).

**Question 1:** Differentiate the following functions:

$$
\tan(x^2 + 3x) \quad (3x \sin(x))^9 \quad \sqrt{\sin(x^2 + 3x)} \quad \left(\sin \left( e^{x^3 + 3x^2} \right) \right)^5.
$$

- Some of the derivatives we can take using the chain rule are so useful, we might want to memorize them separately as shortcuts. (Remember, however, you could always rederive these formulas using the chain rule and other things we know.)

$$
\frac{d}{dx} [(u(x))^n] = n (u(x))^{n-1} u'(x) \\
\frac{d}{dx} [e^{f(x)}] = f'(x)e^{f(x)} \\
\frac{d}{dx}[a^x] = \ln(a) \cdot a^x
$$

Look back to Sept. 10; now we know what the constant is that multiplies $a^x$ when you take its derivative. (It is $\ln(a)$, which is the logarithm of $a$ in the base $e$ because $e^{\ln(a)} = a$.)

- Just like the product and quotient rules, you can apply the chain rule when you only know values of the functions and their derivatives at specific points. However, in the case of the chain rule you would need to know them at different values of the input variable. (Specifically, if you know $g(a)$, $g'(a)$ and $f'(g(a))$ (not $f'(a)$!!) you can compute the derivative of $f \circ g(x)$ at $x = a$.)

**Question 2:** Suppose I know that $f(2) = 5$, $g(1) = 2$, $f'(2) = -3$ and $g'(1) = -8$. Now define $h(x) = f \circ g(x)$. What is $h(1)$? What is $h'(1)$?

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**Homework**

**Web Problems:** The homework assignment named “HW 9/19” is due on Wednesday. Please start it now and ask me about any difficulties you encounter on Monday.