Key Ideas from Sections 2.3 and 2.7

• The rules for algebraic evaluation of limits (Section 2.3) show that limits distribute nicely over sums, products and powers.

• The two most important rules (in my opinion) are these:
  – If \( f(x) \) is a polynomial or a rational function and \( a \) is in the domain of \( f \) then
    \[
    \lim_{x \to a} f(x) = f(a).
    \]
    (This is stated in a box on page 102.)
  – If \( f(x) = g(x) \) for all \( x \)'s except one, then they have the same limit at every point.
    (This appears in a box near the top of page 103.) For example, \( f(x) = \frac{x^2 - 4}{x + 2} \) and \( g(x) = x - 2 \) are equal at every value of \( x \) except \( x = -2 \). So,
    \[
    \lim_{x \to -2} f(x) = \lim_{x \to -2} g(x)
    \]
    and using the previous “important rule” we can see right away that this is \(-4\)!

  ➤ **NOTE:** These rules are not “all powerful”. They help with certain simple examples, but nothing we’ve learned so far can allow us to exactly compute a limit like
    \[
    \lim_{x \to 0} \frac{\sin(5x^2 + 3x)}{x}
    \]
    (which we estimated as being \(3\) in class last time). Towards the end of the course we will learn a method for evaluating these limits exactly.

• **Slopes and Graphs:** Using the precalculus mathematics you already know, you can easily find the slope of the straight line passing through two points on the graph of a function. (We call that a secant line to the graph.) In general, we can say that if \( x \) and \( a \) (with \( x \neq a \)) are both in the domain of the function \( f \) then
    \[
    m = \frac{f(x) - f(a)}{x - a}
    \]
    is the slope of the line connecting the corresponding points on the graph. If \( x \) is very close to \( a \), then this will also be close to the slope of the tangent line at \((a, f(a))\)...but probably would not be equal to it since zooming in enough would reveal to us that \((x, f(x))\) does not quite lie on the tangent line to the graph at \(a\). Still, making \( x \) even closer to \( a \) would make the value of \( m \) even closer to the slope of the tangent line. Unfortunately, we cannot put \( x = a \) into the formula because of the denominator...this is a perfect application for the limit!

• Using these algebraic methods for evaluating limits, we can now find the slope of the tangent line *exactly*. 
Definitions 1 and 2, p. 144-145: The tangent line to the curve \( y = f(x) \) at \((a, f(a))\) is the line through that point with slope

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

This same slope can be computed as

\[
m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Of course, it is only if \( f(x) \) is a simple function that we can figure out the value of this limit using what we know so far...but our skills will improve quite soon.

**Question 1:** Find the slope of the tangent line to \( y = x^2 \) at \((1, 1)\) and the slope of the tangent line to \( y = 3/x \) at \((3, 1)\).

- After we have found the slope of the tangent line, we can also write the equation of the tangent line since we already know the coordinates of a point on it. It is then possible to check that you’ve done everything correctly by “zooming in” on a simultaneous graph of the function and the tangent line at the point of tangency to see that the two become indistinguishable.

**Question 2:** Find the equations of the two tangent lines from the previous question and graph them together with the original functions to verify your answer.

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Homework

**Web Problems:** The online homework assignment labeled “HW 9/1” must be completed by Friday. Most of the problems are from Section 2.3, but the last two are from 2.7 even though we have not fully covered the material from that section.

**Written Homework:** Please complete problem #4 from page 150 to be handed in on Friday. (Note that it is very similar to the last two web problems.) Remember that your goal here is not simply to get the right answer, but to concentrate on how you are communicating the information to me. You may use both words and mathematical symbols, as long as you use them correctly. For part (c), you should also include three sketches of what you see on your calculator with captions that tell me the viewing window and describe any significant features of the figure.

**Hint:** Definition 1 (on page 144) and Equation 2 (on 145) are essentially equivalent. The book asks you to use both just so you can see that they are the same.