Key Ideas

The Fundamental Theorem of Calculus (continued):

- The most basic way to understand the FTC is the use of Part II in evaluating integrals. This is the part most students seem to remember. Before FTC, computing areas or volumes of strangely shaped objects was a task that only super-geniuses like Archimedes could undertake. Thanks to FTC, it is as easy as anti-differentiation.

**Question 1:** Find the exact area of the region under the graph of the function \( f(x) = \frac{1}{x} \) on the interval \([1, 3]\).

**Question 2:** Evaluate \( \int_0^1 (9x^2 - 3) \, dx \) and interpret the answer geometrically.

> Warning: Don’t get over-confident! Remember that anti-differentiation itself can be hard. For instance, the FTC Part II does not help at all with this question:

**Question 3:** Find (or approximate) the area under the arch of the function \( f(x) = \sin(x^2) \) for \( 0 \leq x \leq \sqrt{\pi} \). (Note: We do not know a formula of a function whose derivative is \( \sin(x^2) \)!)  

- Here's another way to think of the FTC Part I (the part that most students have trouble with). Suppose you have a function \( f(x) \) and you want to know a value of one of its antiderivatives. In particular, suppose you want to know \( F(b) \) for the function \( F(x) \) that satisfies \( F'(x) = f(x) \) and \( F(a) = C \). Then, according to the second (easier part) of the FTC you know that

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

The first part of the FTC merely says that if you do know the integral on the left somehow, then you can find \( F(b) \) by the formula

\[
F(b) = F(a) + \int_a^b f(x) \, dx.
\]

**Question 4:** If \( F(x) \) is the function such that \( F'(x) = \cos(x^2) \) and \( F(.5) = 8 \), then use your calculator to find approximate values for \( F(1) \), \( F(1.5) \) and \( F(2) \).

- The preceding example demonstrates a geometric way to think of the FTC. In particular, on an interval where \( f \) is positive, its antiderivative will go up by an amount equal to the area of the region bounded by the graph. Similarly, if it is negative on an interval, then its antiderivative will decrease by an amount equal to the area of the region bounded by the graph. And, in general, the change in the value of an antiderivative on an interval is equal to the integral of the function on that interval.
Suppose we define the function

\[ g(x) = \int_{a}^{x} f(t) \, dt \]

based on a given (continuous) function \( f(x) \). Then we know two things about \( g(x) \): \( g(a) = 0 \) and \( g'(x) = f(x) \). In some homework questions your understanding of the second of these is tested in the most fundamental way by asking you to compute derivatives of functions constructed in this way. If it really says \( \int_{a}^{x} \) then these are super easy (the trick is that you should not make it harder than it is), but in some the endpoints are different.

- If the \( x \) appears in the lower endpoint, then you should use the fact that \( \int_{a}^{b} = -\int_{b}^{a} \) to rewrite it with the variable in the top before differentiating.
- Perhaps the most confusing twist that can be added is then the endpoint of integration is not just \( x \) but a function of \( x \):

\[ h(x) = \int_{a}^{p(x)} f(t) \, dt. \]

In that case, \( h(x) = g(p(x)) \) (a composition) and so you can use the chain rule to see that \( h'(x) = g'(p(x))p'(x) = f(p(x))p'(x) \).

**Homework**

**Web Problems:** A few WebAssign problems are due Thursday.

**Hand-drawn Graph Due Friday:** Problem #1 on the homework asks you to draw a graph. Note that by FTC I this is an antiderivative of \( f(x) \) and so you should make use of all of the stuff you remember about sketching antiderivatives as well as the new material on integrals.