Key Ideas from Section 5.2: The Definite Integral

- This is a long handout. I expect that it will take us at least two lectures to get through it. Consequently, the homework assignment at the end is due on Monday instead of on Friday as you would normally expect.

- **Sigma Notation:** You would think that we already have a notation for adding up numbers ("+") and don’t need another...but you’d be wrong. It is very useful to have a way to write down a sum of a bunch of terms if you have a formula for each. We use the notation

\[ \sum_{i=m}^{n} h_i = h_m + h_{m+1} + h_{m+2} + \ldots + h_{n-1} + h_n. \]

Thus, for instance, \( \sum_{i=1}^{100} i^2 \) is a convenient shorthand way to write the sum of the squares of the first one hundred positive integers.

- **Rightsums, Leftsums and Midsums:** Given a function \( f(x) \) and an interval \([a, b]\), divide \([a, b]\) into \(n\) pieces each of length \( \Delta x = \frac{b-a}{n} \). Call the endpoints of these intervals \( x^*_0, \ldots, x^*_n \). Also, we name the midpoints of the intervals \( \bar{x}_i = \frac{(x^*_i + x^*_{i+1})}{2} \). Then, using sigma notation we define the rightsum, leftsum and midpointsum with \( n \) partitions to be the numbers

\[ R_n = \sum_{i=1}^{n} f(x^*_i) \Delta x \quad L_n = \sum_{i=0}^{n-1} f(x^*_i) \Delta x \quad M_n = \sum_{i=1}^{n} f(\bar{x}_i) \Delta x. \]

Any one of these is called as Riemann sum.

- **The Connection to Last Lecture:** If \( f(x) \) happens to be positive on the interval \([a, b]\), then this is the same thing we did last time to approximate areas. Specifically, note that \( f(x^*_i) \Delta x \) is the area of a rectangle of width \( \Delta x \) and height \( f(x^*_i) \) and so \( L_n, R_n \) and \( M_n \) should be estimates of the area under the graph. However, we are not requiring that \( f(x) \) be positive anymore...these numbers are defined in any case.

- **An Important Observation and Definition:** Because of the continuity of \( f \), the limits of the left, right and midpointsums as \( n \) goes to infinity are equal

\[ \lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n = \lim_{n \to \infty} M_n. \]

So we define the integral to be this number!

Given a function \( f(x) \) that is continuous on the interval \([a, b]\) we define the definite integral

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} R_n \]

to be the limit of the rightsums as \( n \) goes to infinity. (It would be just as good to let it be the limit of the leftsums or midpointsums since they give the same thing.)
• **Geometric Interpretation:** If \( f(x) \) has only positive values on \([a, b]\) then we know that \( \int_a^b f(x) \, dx \) is the area between the graph and the \( x \)-axis. If \( f \) takes any negative values, then things are a little more complicated. In general, we can say that

\[
\int_a^b f(x) \, dx = \left( \text{total area above } x\text{-axis and below graph on interval } [a, b] \right) - \left( \text{total area below } x\text{-axis and above graph on interval } [a, b] \right)
\]

• **Important:** Note that if \( f(x) \) is not always positive, this is not the total area of the region bounded by the graph. In particular, it can be zero (if the area over \( x \)-axis and area below \( x \)-axis are the same) or even negative (if the area below is bigger than the area above).

We can use this fact in two different ways: use knowledge of areas to evaluate integrals or vice versa.

• **How To Compute Riemann Sums:** If \( n \) is small (no bigger than 5 say), you should be able to compute \( L_n, M_n \) or \( R_n \) by hand without too much trouble. In fact, you will definitely have to do this on the next test and probably on the final, showing your computations. However, it is also possible to do these on a calculator or computer. The main advantage is then you can do the case of really large \( n \)'s without too much trouble.

**On the Internet:** Use the Riemann Sum Calculator (Java program) I wrote at [http://kasmana.people.cofc.edu/MATH120/rsums.html](http://kasmana.people.cofc.edu/MATH120/rsums.html). (You need to use the right notation. For instance, note that you must type \( \text{power}(x,2) \) for \( x^2 \).)

**With your Calculator for given \( n \):** You need to know what \( f(x), \Delta x \) and the \( x^*_i \) are and then you can find the value of \( R_n \) or \( L_n \) on your calculator using commands under the LIST OPS menu:

\[
\begin{align*}
R_n &= \text{sum(seq}(f(x)\Delta x, x, x^*_1, x^*_n, \Delta x) \\
L_n &= \text{sum(seq}(f(x)\Delta x, x, x^*_0, x^*_{n-1}, \Delta x)
\end{align*}
\]

Similarly, you can find \( M_n \) as

\[
M_n = \text{sum(seq}(f(x)\Delta x, x, \frac{x_1+x_n}{2}, \frac{x_n-x_1}{2}, \Delta x)
\]

**With your Calculator if you don’t care about \( n \):** Your calculator actually has a neat feature for easily estimating integrals by a Riemann sum, but it doesn’t tell you what \( n \) is! When looking at a graph of \( f \), select MATH and then the symbol that looks like \( \int f(x) \). It even shades the region between the graph and the \( x \)-axis, but remember it is just an approximation!

• It is useful to realize that you don’t need to know a formula for \( f(x) \) to compute its Riemann sum if you are given the values of the function at all of the necessary points. (See, for example, problem 2 in the homework.)

• Note that if \( f(x) \) is an increasing function then \( R_n \) is always an over-estimate of an integral and \( L_n \) is always an under-estimate. If \( f(x) \) is a decreasing function then it is the other way around. If it is sometimes increasing and sometimes decreasing then you cannot say very easily which is too big and which is too small. In particular, if you consider this example:

\[
\int_{-1}^{1} \sqrt{1 - x^2} \, dx
\]

you will see that \( L_n \) and \( R_n \) are equal to each other, but neither is equal to the actual integral!

• Some useful properties of integrals:
\[
\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \\
\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \\
\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \\
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \\
\int_a^a f(x) \, dx = 0 \\
\int_a^b f(x) \, dx = \int_a^b f(t) \, dt
\]

- **Summary – how to compute/estimate definite integrals**: So far, we only have four ways to determine a value of a definite integral, and all but one of them only gives an approximate value.

1. **Exact Area Computations**: If the region bounded by the graph is made up of triangles, rectangles and semi-circles, then we can use our knowledge of basic geometry to determine the areas and then the integral. This is the idea in questions 6–10 (and 14, sort of).

2. **Left, Right and Midsums by Hand**: As explained in the last lecture, you can compute \( R_n, L_n \) and \( M_n \) by breaking the interval into \( n \) pieces and then multiplying the width of these pieces by the sum of the the values of \( f \) at the appropriate point (left endpoint, right endpoint or midpoint). This is of fundamental importance because it is the actual definition of the integral and because it is the basic way in which integrals are computed when shortcuts are not available. This is what you are expected to use in problems 1–5.

3. **Left, Right and Midsums using a Calculator**: As explained above, you can use your calculator or my Java program to evaluate left, right and midsums for any \( n \). This is useful for recognizing that the values converge as \( n \to \infty \) and again because this is the only method to use when no shortcuts are available. Problems 11, 12 and 14 use this method.

4. **Your Calculator’s Best Estimate**: Graphing calculators can generally approximate integrals using a built-in feature which is (basically) just left, right and midsums with a really large \( n \). I don’t like the fact that the method used to compute this is hidden from you, but it is in other ways a useful feature. Only problem 13 in your homework is based on this, but you can use it to check whether your other answers make sense at any time!

**Question 1**: Find the exact value of \( \int_0^3 (x - 2) \, dx \) using geometry.

**Question 2**: Estimate the value of \( \int_0^1 3 + \sqrt{1 - x^2} \, dx \) with the approximation \( R_4 \). Compute a better estimate with your calculator. Then, determine the exact value geometrically.
The Real Definition: Once again, don’t forget that the true definition of the integral is

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} R_n. \]

In other words, it is the number you get when you find the right sum (or left or midsum if you prefer) for larger and larger values of \( n \). In fact, the notation for the definite integral mimics the notation for the rightsum.

Homework

Web Problems: Homework set “Section 5.2 - 4/6” is due Monday.