Key Ideas from Section 4.7

**Optimization:** making some real world quantity as big (or small) as possible

- Recall from February 21st (Section 4.1) that a function may have an *absolute maximum or minimum* value on an interval. Moreover, if the interval is a closed interval and the function is continuous on it then we have a relatively simple procedure for finding those absolute extrema!

  Well, there are many “real world” situations in which this is exactly what you want to do. (In fact, the big companies make use of mathematicians for exactly this sort of reason. See the later in this handout for some examples.)

- The calculus here is not very difficult: once you have the function $f(x)$ that you want to max/minimize and the interval $I$, you do one of two things. If $I$ is a closed interval and $f$ is continuous, you use the method described in Section 4.1 (compare values of $f$ at the endpoints and all critical points in $I$). If $I$ is not closed or $f$ is not continuous on the interval, you must use the sketching techniques we learned (Section 4.5) to make a sketch of $f$ on the interval $I$. In the end, your goal is to identify the biggest (or smallest) value that the function takes on the interval.

- The hard part is getting to that point. If a situation is described to you, you may need to figure out a formula for the function you want to optimize and you may need to figure out the interval on which to optimize. [For example, to optimize the profit function $P(x)$ depending on the price $x$ that you charge for your product, it is sufficient to consider $0 \leq x \leq b$ where $b$ is any price that is so expensive that nobody will buy your product.]

- One difficulty that often arises is that it may look as if there are too many variables in the function. (It may be a function of two, three or five hundred variables.) This can happen in the real world, but not in this class! There must be some restriction (a constraint) which will allow you to eliminate all but one variable.

**Question 1:** Suppose you want to build a fence around a rectangular region. Three of the sides can be made of a material that costs $5 per foot, and one side must be made of a stronger material that costs $15 per foot. What are the dimensions of the biggest rectangle you could fence in this way if you have a total of $2000 to spend on the fence?

**Question 2:** Suppose we have a piece of wire 12 inches long. We want to cut it into two pieces one of which will be bent into a square and the other into a circle. Where should the cut be made so that the total area contained in the square and the circle are as small as possible?

**Answer:** Let us call “$x$” the length of the piece that we will make into the circle. Then, the square is going to be made out of the piece that is $12 - x$ inches long and each of the four sides will be $(12 - x)/4$ inches long and so the area in the square will be $((12 - x)/4)^2$. Similarly, the circle will have a circumference of $x$ inches and since $C = 2\pi r$ the radius will
be \( x/(2\pi) \) and the area of the circle will be \((x/(2\pi))^2\pi\). Putting this together we see that the total area will be

\[
A(x) = \frac{x^2 - 24x + 144}{16} + \frac{x^2}{4\pi}.
\]

We want to find the minimum value that this function takes on the interval \( 0 \leq x \leq 12 \). We will check the endpoints (which would correspond to making just a circle or just a square without cutting), but we also need to check any critical points in the interval. We find \( A'(x) = \frac{2x-24}{16} + \frac{x}{2\pi} \) and the critical point is \( x = \frac{12\pi}{4+\pi} \).

To determine which of the three “suspects” corresponds to the minimum value of the area function, we just compare

- \( A(0) = 9 \)
- \( A\left(\frac{12\pi}{4+\pi}\right) \approx 5.04089 \)
- \( A(12) = \frac{36}{\pi} \approx 11.4592 \).

The critical point does indeed correspond to the minimum!

- **Warning:** The critical point is not always the answer. (A famous story relates how an airplane designer redesigned the wings using calculus to minimize the turbulence...but he had just picked the critical point and assumed it was what he wanted. In fact, it *maximized* the turbulence instead!)

**Question 3:** What would the answer to the previous question be if we had wanted to *maximize* the area instead of minimize it?

- When you do an “optimization problem” on an exam, project, or homework, your work must demonstrate that your answer really is the absolute maximum. (That is, it is not enough to correctly find the absolute maximum without showing that it really is the max.)

**Question 4:** We want to build a rectangular box with no top and square ends with a volume of exactly 100 cubic inches. What dimensions will minimize the surface area? (Note: This one cannot be done with the "closed interval method". How can you verify that you indeed have the minimum and not the maximum?)

**Question 5:** (A tricky one.) Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 1 kilometers west and \( \frac{1}{2} \) kilometer north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 km/hour. She can also travel through the grass in the park, but only at a rate of 4 km/hour (the park is a favorite place to walk dogs, so she must move with care). What path will get her to the bus stop fastest?

**Answer:** She will want to walk straight along the southern edge of the park for a while, and then cut across the park at some point to head straight for the bus stop. Let \( x \) denote the number of kilometers she will walk along the edge before cutting through. Then, it will take her \( x/6 \) hours to walk that far. We can use the Pythagorean Theorem to determine that her walk through the grass will be \( \sqrt{\frac{1}{4} + (1 - x)^2} = \sqrt{x^2 - 2x + 5/4} \) kilometers long and so it will take her \( 1/4\sqrt{x^2 - 2x + 4/5} \) hours to walk it. All together, it will take her

\[
T(x) = \frac{x}{6} + \frac{\sqrt{x^2 - 2x + 4/5}}{4} \text{ hours.}
\]

It would be silly for her to walk away from or past the bus stop, so we know we need only consider \( 0 \leq x \leq 1 \), but those end points really are possible solutions. (It could be fastest
for her to just head straight for the bus stop from the start, or to walk along the edge until she is just south of the stop.)

To find critical points we find $T'(x)$ and set it equal to zero. It takes some complicated algebra, but we eventually determine that $x = \frac{5 - \sqrt{5}}{5} \approx 0.552786$ is the only critical point in the interval $0 \leq x \leq 1$. Now, we have three “suspects”, this critical point and the two endpoints. We just need to find out how long each would take her and choose the one that is shortest! For this, it is sufficient to use a calculator to get an estimate:

$$T(0) = \frac{\sqrt{5}}{8} \approx 0.279 \text{ hrs} \quad T\left(\frac{5 - \sqrt{5}}{5}\right) \approx 0.2598 \text{ hrs} \quad T(1) = \frac{7}{24} \approx 0.291667 \text{ hrs}.$$  

There is not a big difference (all between 15 and 18 minutes), but walking exactly $(5 - \sqrt{5})/5$ kilometers before cutting across the park is the path that would take the least amount of time!

**Homework**

**Web Problems:** The homework problems in the set “Section 4.7 - 03/23” are due on Friday evening.

**Real Examples of Optimization: (Someone does these for a living)**

- A dairy farmer gets fresh milk from a cow. He can pull off some percentage $p$ of the milk to sell separately as cream and skim milk, and the rest of the milk is sold as “whole” milk. To maximize his profit as a function of $p$, he has to know the cost of cream, skim milk and regular milk. Since these values are always changing, the dairy industry uses calculus based optimization methods to recompute the value of $p$ that maximizes profit on a regular basis.

- The Coca-Cola Company sells billions of cans of soda each year. They could make the can a little bit taller and a little bit narrower and it would hold the same amount of soda, but have a different surface area. Is it possible that the surface area would get smaller if they made this change? If this was true, they could save a fraction of a penny per can in aluminum...which would be millions of dollars per year. Don’t you think they would first check to make sure that they have minimized the cost of the can? (See page 333 for a more detailed discussion of this example.)

- As we have already mentioned a few times, a manufacturer can affect their profits by changing how much they charge for their product. Increasing the price may increase the profits, even though it will decrease sales. But, increasing the price too much will eventually lead to no sales, which is zero profit. So, there is a price that can be charged to maximize profits on the closed interval between the price being zero and the price being so high that nobody buys the product. If the difference between the optimum profit and the profit you are currently getting is likely more than $100,000 per year, it is worthwhile to hire a quantitative analyst (a mathematician in the business sector) to ensure that you are getting the maximum!

- In one of my favorite examples, a mathematical model was used to figure out the best dosage of the drug AZT to give to AIDS patients. The number of healthy T-cells in the patients body was modeled as a function of the amount of drug that they were given (as well as other variables like the frequency with which the drug was given). Calculus based methods were used to optimize the situation: what dosage leads to the greatest number of healthy T-cells?