Key Ideas from Section 4.3

I hope you had a good break!: We will pick up right where we left off, in Section 4.3.

• To determine where \( f(x) \) is increasing and where it is decreasing:

1) On a number line mark each value where \( f'(x) = 0 \) or DNE. (These marked points are the only places that \( f \) can change its direction.)

2) Determine the sign of \( f'(x) \) on each interval “cut” by the marked points. (One way to do this is to simply pick some number \( c \) in that interval and check the sign of \( f'(c) \).)

3) On any interval where \( f'(x) > 0 \), \( f \) is increasing. On any interval where \( f'(x) < 0 \), \( f \) is decreasing.

• The First Derivative Test for Local Extrema: If \( c \) is a critical number for \( f \) and \( f \) is continuous at \( x = c \), check to see if \( f'(x) \) changes signs at \( c \).

\[
\begin{align*}
+ & \quad \rightarrow \quad - \quad \Rightarrow \quad \text{local max at } x = c \\
- & \quad \rightarrow \quad + \quad \Rightarrow \quad \text{local min at } x = c \\
\text{no sign change} & \quad \Rightarrow \quad \text{not a local extremum}
\end{align*}
\]

Question 1: Basic Example: On what intervals is \( f(x) = x^4 - \frac{8}{3}x^3 + 2x^2 - 5 \) increasing and on what intervals is it decreasing? Identify any local extrema.

Remember: Not every critical point is a local extremum.

Question 2: Trickier Example: On what intervals is \( f(x) = 3x^{2/3} - 2x \) increasing and on what intervals is it decreasing? Identify any local extrema?

Remember: Critical points can occur at points in the domain of \( f \) where \( f'(x) \) is undefined.

Question 3: Trickiest Example: On what intervals is \( f(x) = \frac{1 + x^3}{x^2} \) increasing and on what intervals is it decreasing? Identify any local extrema?

Remember: We do not say that \( x = a \) is the location of a local extremum of \( f \) when \( a \) is not even in the domain of the function!

• Concavity: “Concave up” means that the slopes of \( f \) increase as you move right. This is the same as saying that \( f' \) is increasing or that \( f'' \) is positive! “Concave down means that the slopes of \( f \) are decreasing as you move to the right. This is the same as saying that \( f' \) is decreasing or that \( f'' \) < 0. [Example: the graph of \( y = x^2 \) is concave up everywhere and the graph of \( y = -x^2 \) is concave down everywhere.]

• Notice that concavity is exactly what we were looking at when trying to decide if the linear approximation is an overestimate! If \( f''(a) < 0 \) then the linear approximation near \( x = a \) gives an overestimate.
• A point on the graph where the concavity changes is called an **inflection point**. You can find it (just like finding local extrema) by checking to see where $f''$ changes sign. (Note: an inflection point of $f$ is a local extremum of $f'$.)

**Question 4:** Find the exact $x$ and $y$-coordinates of the inflection points on the graph of the function $f(x) = x^4 - \frac{8}{3}x^3 + 2x^2 - 5$ from Question 1 above.

- There is another way to test for local extrema. It is different than the First Derivative test from above. There are certain situations where each one is “the best choice”.

• **Second Derivative Test:** If $c$ is a critical number for $f(x)$ and
  - $f''(c) > 0$ then $f$ has a local min at $x = c$
  - $f''(c) < 0$ then $f$ has a local max at $x = c$

  Warning: if $f''(c) = 0$ or is undefined then you cannot conclude anything from this test.

**Question 5:** The function $f(x) = x^4 - 14x^2 + 24$ has a critical point at either $x = 1$ or $x = -1$. Which is it? Use the second derivative test to identify whether it is a local max or local min.

**Question 6:** The function $f(x) = x^4$ has a critical point at $x = 0$. Which test can be used to identify whether it is a local max or local min?

• Note: if I tell you a formula for $f'$ or give you its graph, you can find the local extrema, inflection points, and the intervals on which $f$ is concave up/down, increasing/decreasing without having to know $f$ at all!

**Question 7:** A function $f(x)$ has derivative $f'(x) = x(x - 2)^2e^x$, whose graph is shown in the figure below.

![Graph](image)

On what interval(s) is $f(x)$ increasing? At what values of $x$ does $f(x)$ have a local extremum? How many inflection points are there on the graph of $f$ and approximately where do they occur?

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**Homework**

**Web Problems:** “Section 4.3 - 3/4” is due Wednesday.