Key Ideas from Section 3.5: Inverse Trig Functions

- Recall from our first class that the inverse of a function $f$ is the function that undoes whatever $f$ does to its input number. For instance, if $f(3) = 25$ then $f^{-1}(25) = 3$.

- Recall also that some functions do not have inverses. In particular, if $f(x_1) = f(x_2)$ for $x_1 \neq x_2$ then there is no inverse function. (This is like a “horizontal line test”.) For instance, if $f(3) = f(5) = 25$, then what would $f^{-1}(25)$ be? It can’t be both 3 and 5!

- From this you would conclude that the trig functions $\sin(x)$, $\cos(x)$, $\tan(x)$, etc... do not have inverses since each of them has the same value at $x + 2\pi$ as it does at $x$. In particular, if one was looking for $\cos^{-1}(1)$ you might think that 0 would be a good value for it (because $\cos(0) = 1$)...but so could $2\pi$ since $\cos(2\pi) = 1$ also!

- Since we do often want to know something like “which angle $\theta$ has a sine of .5?”, we get around this difficulty by making a restriction. Even though there are many angles that have a sine of .5, there is only one angle between $-\pi/2$ and $\pi/2$ that does! (This gives us just the right half of the circle, for which no two points have the same $y$-coordinate.)

\[ \theta = \sin^{-1}(x) \Leftrightarrow \sin(x) = \theta \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \]

- Similarly, we can again use the right half of the circle also to talk about an (almost) inverse of the tangent function:

\[ \theta = \tan^{-1}(x) \Leftrightarrow \tan(x) = \theta \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \]

- The function $\tan^{-1}(x)$ is a particularly interesting one because of its graph. For very negative values of $x$ it looks almost like the horizontal line $y = -\pi/2$, and for very positive numbers it looks almost like $y = \pi/2$, but near $x = 0$ it smoothly moves from one to the other. We will learn later that these lines are called “asymptotes” of the graph and it is interesting that this natural function has two different horizontal asymptotes.

- Finally, we can look at just the top half of the circle, where no two points have the same $x$-coordinate to define an (almost) inverse for $\cos(x)$:

\[ \theta = \cos^{-1}(x) \Leftrightarrow \cos(x) = \theta \text{ and } 0 \leq \theta \leq \pi \]

- There are (almost) inverses for the other trig functions, too. See page 70 for a list of them and the corresponding ranges of angles.

- Differentiating Inverse Functions: Implicit differentiation gives us a good way to find the derivatives of the inverses (even “almost” inverses like these) of functions whose derivatives we know. The idea is that we can take the formula $y = f^{-1}(x)$, rewrite it in the form $f(y) = x$, differentiate implicitly (with $y$ as a function of $x$), and then solve for $y'$!
• **Alternative Names:** For some reason, these functions are sometimes also called arcsine, arccosine, arctan, etc. I'm not sure why. **On webassign it seems that you’re expected to call them** \( \text{asin}(x), \text{acos}(x), \text{etc.} \)

• **The Derivative of arcsine:** Let us apply this idea to the equation \( y = \sin^{-1}(x) \) to see what the derivative of inverse sine might be:

\[
\sin(y) = x \Rightarrow \cos(y)y' = 1 \Rightarrow y' = \frac{1}{\cos(y)} = \frac{1}{\cos(y)}
\]

Hmm....that doesn't seem so good because it gives the derivative in terms of \( y \) rather than \( x \). We know that \( \cos(y) \) is a positive number (since we're looking at only the right half of the circle), and so we can use the fact that

\[
\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}
\]

to rewrite the derivative in its nicest form:

\[
\frac{d}{dx} [\sin^{-1}(x)] = \frac{1}{\sqrt{1 - x^2}}.
\]

(Take a look at the graph to make sure it seems reasonable.)

> Personally, I think it is cool that we have a formula for the slope of every tangent line on the graph of \( y = \sin^{-1}(x) \)...and especially that the formula doesn't even involve any trig functions!

• We can similarly find the derivatives of the other inverse trig functions:

\[
\frac{d}{dx} [\cos^{-1}(x)] = -\frac{1}{\sqrt{1 - x^2}} \quad \frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{1 + x^2}.
\]

See page 213 for the derivatives of the other three inverse trig functions.

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**Homework**

**Web Problems:** There are just a few problems on differentiating inverse trig functions in “Section 3.5 - 2/10”. Note that WebAssign likes you to refer to the inverse sine function as “\( \text{asin}(x) \)” (which you can just type or select from the palette), and similarly for the others.

**Written Problem (Reminder):** Answer the following question on a piece of paper to be turned in on Monday. You will be graded on your clarity and exposition, so think about how you are writing it as well as the numerical value which is your final answer:

The graph of the equation

\[
x^2 - y^2 = (3x - 3)y
\]

is shown in the figure to the left. (a) Find a formula which gives the slope \( \frac{dy}{dx} \) at every point \( (x, y) \) on the graph. (b) As you can see in the picture, there are two points on the graph which have \( x \)-coordinate equal to 1. What are the exact slopes of the tangent lines at those two points?