Final Remarks about Numbers, Functions and Limits

Today we get to talk about “the derivative”. Perhaps from the student’s point of view, this means we are finally going to be getting to calculus. However, our time talking about numbers, functions and limits was not wasted. I assure you, the properties of numbers (especially the small differences between “close” numbers), the properties of functions and especially the concept of limits are the keys to understanding and using calculus.

So, before we move on, let me illustrate some key ideas with these examples:

Last time, we looked closely at the function \( f(x) = (1 + x)^{1/x} \). Note that this function is undefined at \( x = 0 \). That means that “\( f(0) \)” is not a number. However, we can still ask what \( \lim_{x \to 0} f(x) \) is! (There is a difference between \( f(a) \) and \( \lim_{x \to a} f(x) \). If you think they are the same, you are missing a key idea.) A homework question asked us to estimate this limit to five decimal places. So, we looked at the following table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>.001</th>
<th>.0001</th>
<th>.00001</th>
<th>.000001</th>
<th>.0000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2.7169239</td>
<td>2.7181459</td>
<td>2.7182682</td>
<td>2.7182805</td>
<td>2.7182817</td>
</tr>
</tbody>
</table>

Notice that even though the function is undefined at \( x = 0 \), as the values of \( x \) get close to zero they do not seem to be doing anything crazy (they are not getting super huge or jumping up and down with no pattern). In fact, we see a pattern in that the digits of the decimal expansion start “settling down”. The first two both start \( 2.71 \) (but then disagree), the next two both start \( 2.7182 \) (but then disagree). What I am looking for (since the question asked for 5 decimal places) is for the first five decimal places to be the same when we round off. This is true for the last two (coming from input values very close to zero). They both round to 2.71828...so that is the answer the computer wanted.

Is this the famous number \( e \)? There is no way to tell from just the first five digits, as the next example illustrates.

Consider \( g(x) = \frac{-355 + 355x - 355x^2 + 355x^3}{226x - 226} \). Note that \( g(1) \) is undefined.

What is \( \lim_{x \to 1} g(x) \)? From the table below you might guess that it is going to be \( \pi \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.001</th>
<th>1.0001</th>
<th>1.00001</th>
<th>1.000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>3.14474</td>
<td>3.14191</td>
<td>3.14162</td>
<td>3.141593235</td>
</tr>
</tbody>
</table>

**HOWEVER**, it is not. The limit is a number which has the same first six digits as \( \pi \) but then has different digits from then on. As I explained on the first day of class, we will be very aware of the small differences between close numbers. The limit of this function \( g \) as \( x \) goes to 1 may be quite close to \( \pi \), but it is not equal to \( \pi \)!
Some New Notation: We will call the slope of the tangent line to $f(x)$ at $x = a$ “the derivative of $f$ at $a$” and write it as $f'(a)$. (Section 2.8). So, $f'(a)$ is given by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

(and we can estimate this as in Section 2.2 or compute it exactly as in Section 2.3)! If we want to know the slope at only one value of $a$, we can substitute this in at the beginning. Or, if we want to know $f'(a)$ for many different values of $a$, we can evaluate this limit keeping $a$ arbitrary.

**Question 1:** Let $f(x) = 1/(1 + x)$. Use the formula above to find $f'(2)$.

Those of you who do not have much interest in tangent lines will be happy to know that the derivative has applications outside of geometry. It comes from thinking of the slope of the secant as an average rate of change, and then the derivative gives a piece of information that is often more important: the instantaneous rate of change. Consider the idea of velocity as a good example.

When you learned how to compute velocity back in elementary school, you were told that the formula was distance/time. If the velocity is constant, this does work. (A train travelling at constant velocity which goes 100 miles in 2 hours was travelling at 50 miles per hour.) But, once velocity is allowed to vary, this fails to be so precise. (I may have taken 2 hours to drive the 100 miles to Columbia, but if I stopped for a bite in Orangeburg on the way then my velocity was definitely not 50 mph the whole time.)

Instead of completely forgetting about the formula from grade school, let’s just say that it gives us the average velocity over that range of time. (That makes sense with the driving to Columbia example: I drove less than 50 mph when I stopped, and more than 50 much of the rest of the time, and the average was 50!) Thus, if $f(t)$ is a function that gives the position of a moving object at time $t$ and $t = x$ and $t = a$ are two times, then $(f(x) - f(a))/(x-a)$ is the average velocity over the time interval from $t = x$ to $t = a$. (Hey, that’s the same formula as the slope of the secant line!)

But then, how do we get the instantaneous velocity (what you see on the speedometer)? Take shorter and shorter intervals of time, until eventually the interval is a fixed instant. (Hey, that’s the same as the slope of the tangent!)

Remember that we saw the slope of a linear function as an “exchange rate” telling us how a change in the input number would affect the output number. The number $f'(a)$ plays this same role for any function, even if it is not linear. So we call it the instantaneous rate of change. Sometimes we write $f'(a)$ as $df/dx$ to emphasize that it represents the ratio of the change in $f$ by the change in $x$. One of the best ways to remember this is to think of the units in which $f'(x)$ is measured: it is always (units of output)/(units of input).

**Physical Application:** If $f(t)$ gives the position of an object at time $t$, then $f'(a)$ gives the instantaneous velocity at time $t = a$. (Velocity is a number whose absolute value is the speed and whose sign indicates direction of motion.) You can understand this either algebraically (write a formula for the average velocity between any two different times, and then move those points closer together) or graphically (draw the tangent line and imagine the
two curves as representing two different moving objects that meet – the coincidence of slope means they are travelling at the same speed).

**Question 2:** Suppose a toy car was \( f(t) = \frac{1}{1 + t} \) feet away from me \( t \) seconds after it starts moving at time \( t = 0 \). How fast is it moving at time \( t = 2 \)?

- **Another example of “exchange rate”:** Suppose that \( C(t) \) is the temperature in degrees Celsius \( t \) hours after midnight (see example 5 on page 152). Then \( \frac{dC}{dt} \) (which is just another way of saying \( C'(a) \) for any \( a \)) tells how quickly the temperature is rising in degrees/hour.

- **An example without time:** Let \( P(x) \) be the price in dollars of \( x \) gallons of blue paint. If we know that \( P'(10) = 2 \), that means that if you’re already buying 10 gallons, you can buy a little bit more at $2/gallon. If this were a linear function, this “exchange rate” (slope) would remain constant. However, since prices go down when purchasing in bulk, it could be that \( P'(100) = .75 \) (an extra gallon only costs 75 cents if you’re already buying 100 gallons)!

---

**Homework**

**Web Problems:** Do the assignment called “Section 2.7 - 1/21” which is due on Monday. The problems all come from section 2.7 in the book.

**Hint:** In some of these problems there is no computation. You just need to demonstrate your understanding of the relationship between the slope of the tangent line of a graph and the instantaneous rate of change of the function.