Bispectrality and Duality

Alex Kasman, College of Charleston
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What is *bispectrality*?

**Spectral Parameter**

We often consider *families of eigenfunctions* for Lax operators for which the eigenvalue depends upon an extra parameter. For example

\[
L = \partial_x^2 - \frac{2}{x^2} \quad \psi(x, z) = \left(1 - \frac{1}{xz}\right) e^{xz} \quad L\psi = z^2\psi. \quad (*)
\]

**Bispectrality**

In some cases, the same eigenfunction satisfies a pair of eigenvalue equations with the roles of spatial and spectral parameters switched.

**Definition:** \((L_x, \Lambda_z, \psi(x, z))\) is a bispectral triple if

\[
L\psi(x, z) = p(z)\psi(x, z) \quad \text{and} \quad \Lambda\psi(x, z) = \pi(x)\psi(x, z).
\]

**Example:** \(L = \partial^2, \Lambda = z\partial_z, \psi = z^x: \quad L\psi = (\ln z)^2\psi \quad \Lambda\psi = x\psi\)

**Example:** Example \((*)\) above is trivially bispectral since \(\psi(x, z) = \psi(z, x)\).
Bispectrality: Schrödinger Case

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**Theorem 0.1.** The potentials \( V \) for which (0.1), (0.2) hold (for non-zero \( \phi \) and \( A \) of positive order) are \( V(x) = ax + b, \ a, b \in \mathbb{C}, \ a \neq 0 \) (Airy) or \( V(x) = \frac{c}{(x-a)^2} + b \), \( a, b, c \in \mathbb{C} \) (Bessel) or, modulo a translation in \( x \) and adding a constant to \( V \), those which can be obtained from \( V = 0 \) or \( V = -\frac{1}{4} \frac{1}{x^2} \) by finitely many rational Darboux transformations.
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Theorem 0.1. The potentials \( V \) for which (0.1), (0.2) hold (for non-zero \( \phi \) and \( A \) of positive order) are \( V(x) = \alpha x + \beta, \alpha, \beta \in \mathbb{C}, \alpha \neq 0 \) (Airy) or \( V(x) = \frac{c}{(x-a)^2} + b, a, b, c \in \mathbb{C} \) (Bessel) or, modulo a translation in \( x \) and adding a constant to \( V \), those which can be obtained from \( V = 0 \) or \( V = -\frac{1}{4} x^2 \) by finitely many rational Darboux transformations.

This is interesting because it shows that the question is not trivial. E.g. the potential must be rational function.
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More interesting: can be summarized by saying the potential must be a rational KdV solution! (Dynamics or coincidence?)
Bispectrality: Rank 1 Case

- G. Wilson (1993) completely characterized the set of all bispectral triples \((L, \Lambda, \Psi(x,z))\) where \(L\) commutes with other odos of relatively prime order. (Answer: iff spectral curve is rational with only cuspidal singularities.)

- Wilson made use of the known correspondence between such operators with solutions to the KP equation.

- Turns out that \(\Lambda\) commutes with relatively prime order too, so we have \(\Psi(x,z) \leftrightarrow \Psi(z,x)\) (bispectral involution)
What is Classical Duality?

What is a particle system?
Consider the positions $x_i$ and momenta $y_i$ of $n$ particles as functions of time $t$. The Hamiltonian function $H(x_1, \ldots, x_n, y_1, \ldots, y_n)$ determines their dynamics according to

$$\frac{\partial x_i}{\partial t} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial y_i}{\partial t} = -\frac{\partial H}{\partial x_i}.$$ 

What is integrability?
Only for rare choices of $H$ are there explicit $x_i(t)$ and $y_i(t)$. In those cases, there is a function ("symplectic map") $F : (x_i, y_i) \rightarrow (X_i, Y_i)$ such that $\dot{X}_i = 0$ and $\dot{Y}_i$ are constant.

In essence, this $F^{-1}$ takes simple "linear" dynamics and twists it into a complicated looking rule.

Duality:
Integrable particle systems have a natural "duality": pair the system with linearizing map $F$ with the one that has $F^{-1}$!
What is Classical Duality?
“CARTOON”

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that takes simple “linear” dynamics in 2N-dimensional space to complicated looking rules in 2N-dimensional space.

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Consider the positions $x_i$ and momenta $y_i$ of $n$ particles as functions of time $t$. The Hamiltonian function $H(x_1, \ldots, x_n, y_1, \ldots, y_n)$ determines their equations of motion:

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What is integrability?

If there are explicit $x_i(t)$ and $y_i(t)$, then there is a linearizing map $F: (x, y) \rightarrow (X, Y)$ and its inverse $F^{-1}$. In those cases, there is a function ("symplectic map") $F$ such that \[ \dot{X}_i = 0 \] and \[ \dot{Y}_i \] are constant.

In essence, this $F^{-1}$ takes simple "linear" dynamics and twists it into a complicated rule.
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The Hamiltonian function \( H(x_1, \ldots, x_n, y_1, \ldots, y_n) \) determines the individual positions and momenta of the particles.

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\frac{\partial x_i}{\partial t} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial y_i}{\partial t} = -\frac{\partial H}{\partial x_i}.
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Only if there are explicit \( x_i(t) \) and \( y_i(t) \) are there explicit \( x_i(t) \) and \( y_i(t) \).

For integrable systems, there is a function ("symplectic map") \( F : (x_i, y_i) \rightarrow (X_i, Y_i) \) that takes the simple "linear" dynamics and twists it into a complicated rule.

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Example: Calogero System

In the early 1970’s, F. Calogero showed that the Hamiltonians

\[ H_k = \text{tr} M^k, \quad M_{ij} = y_i \delta_{ij} + \frac{1 - \delta_{ij}}{x_i - x_j} \]

are integrable. This system is known to govern pole dynamics for soliton equations. Its quantum analogue shows extreme exclusion statistics.

Interestingly, J. Moser showed that their linearizing map is an *involution*. This system is *self-dual!*

(Non-self dual example: Ruijsenaars-Schneider is dual to *hyperbolic* Calogero-Moser.)
Quantum Duality = Bispectrality
Quantum Duality=Bispectrality

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Quantum Duality=Bispectrality

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It is remarkable that the BA function turns out to be symmetric with respect to $k$ and $x$. For Coxeter configurations this property has been established in [5].

**Theorem 2.3.** Baker-Akhiezer function $\psi(k, x)$ is symmetric with respect to $x$ and $k$: $\psi(k, x) = \psi(x, k)$. 
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- But, is there any reason to expect to see bispectrality in classical duality?
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My 1995 CMP Paper

Calogero State \( \psi(x,z) \)

Wilson's \( \psi(z,x) \)

Linearization

Bisp. Inv.
Consider the *two* involutions I have mentioned so far.
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Krichever correspondence between Calogero particles and rational KP solutions in 1978: motion of poles of rational solns (no mention of bispectrality or duality).
My 1995 CMP Paper

- Consider the two involutions I have mentioned so far.
- Krichever correspondence between Calogero particles and rational KP solutions in 1978: motion of poles of rational solns (no mention of bispectrality or duality).
- I (unjustifiably) felt clever when I showed that this diagram commutes:
Bispectral KP Solutions and Linearization of Calogero–Moser Particle Systems

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Abstract: Rational and soliton solutions of the KP hierarchy in the subgrassmannian $Gr_1$ are studied within the context of finite dimensional dual grassmannians. In the rational case, properties of the tau function, $\tau$, which are equivalent to bispectrality of the associated wave function, $\psi$, are identified. In particular, it is shown that there exists a bound on the degree of all time variables in $\tau$ if and only if $\psi$ is a rank one bispectral wave function. The action of the bispectral involution, $\beta$, in the generic rational case is determined explicitly in terms of dual grassmannian parameters. Using the correspondence between rational solutions and particle systems, it is demonstrated that $\beta$ is a linearizing map of the Calogero-Moser particle system and is essentially the map $\sigma$ introduced by Airault, McKean and Moser in 1977 [2].

1. Introduction

Among the surprises in the history of rational solutions of the KP hierarchy (and the PDE’s which make it up) are the existence of rational initial conditions to a non-linear evolution equation which remain rational for all time [1, 2], that these solutions are related to completely integrable systems of particles [2, 6, 7], and that a large class of wave functions which have been found to have the bispectral property turn out to be associated with potentials that are rational KP solutions [3, 16, 17]. Within the grassmannian which is used to study the KP hierarchy, the rational solutions, along with the $N$-soliton solutions, reside in the subgrassmannian $Gr_1$ [13]. This paper develops a general framework of finite dimensional grassmannians for studying the KP solutions in $Gr_1$ and then applies this to the bispectral rational solutions. New results include information about the geometry of KP orbits in $Gr_1$ and identification of properties equivalent to bispectrality. In addition, an explicit description of the bispectral involution in terms of dual grassmannian coordinates leads to the conclusion that it is, in fact, essentially the linearizing map $\sigma$ [2].

1 Research supported by NSA Grant MDA904-92-H-3032
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**Contained an important technique:** rank one operator identities.
Bispectrality=Duality Program Overview

Quantum

Classical
In joint paper with Emil Horozov (1998) *used* bispectrality/duality correspondence to produce new dual quantum Hamiltonian pairs from any given example.
Bispectrality = Duality Program Overview
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Quantum

Classical

Calogero Self-duality = bispectrality of scalar ODOs which commute with others of relatively prime order (Kasman ’95 / Wilson ‘98)
Bispectrality=Duality Program Overview

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Not relatively prime

∧ not ODO

Matrix ODOs
Produced rank \( r > 0 \) bispectral rings, poles motion under KP flow is linearized by bispectral involution, and that quantum Hamiltonians are bispectral (Kasman-Rothstein ‘97–’01).

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Bispectrality=Duality Program Overview

I was planning ahead: Bispectrality for Solitons (translation operators) in 1998, Rank One Formulas for Solitons (w/Gekhtman) in 2001...working towards duality

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Matrix ODOs
In 2007, Haine used these matrices and operators to show that indeed the bispectral involution for KP solitons is the action-angle map for Ruijsenaars/Hyperbolic Calogero. (Non-self dual case!)

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  - Matrix ODOs
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- Not relatively prime
- $\wedge$ not ODO
- Matrix ODOs

Old paper of Zubelli suggests that bispectrality and matrices don’t mix. So, nobody looked at it much after.
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A NEW “SPIN” ON PARTICLE DYNAMICS

In addition to position and momentum, the interaction between each pair of particles $i$ and $j$ now depends on the number $s_{ij} = \alpha_i \cdot \beta_j$ where $\alpha_i$ and $\beta_j$ are $r$-vectors. (We assume $s_{ii} = s_{jj}$.)

Note that $R = (s_{ij})$ is a matrix of rank $r$. (In hindsight, the “rank one conditions” from many of my previous papers was a “spinless” assumption.)
Spin Calogero

Let

\[ X_{ij} = x_i \delta_{ij}, \quad Z_{ij} = y_i \delta_{ij} + (1 - \delta_{ij}) \frac{s_{ij}}{x_i - x_j}. \]

The eigenvalues dynamics of \( X + k t Z^{k-1} \) are governed by \( H_k = \text{tr} Z^k \).

Note that the rank \( r \) condition “\([X, Z] - I = R\)” holds.

More generally,

\[ s\text{CM}^n_r = \{(X, Z, A, B) \mid X, Z \in M_{n \times n}, A, B^\top \in M_{r \times n}, [X, Z] - I = BA \neq 0\} \]

is the state space of the spin Calogero system (including particle “collisions”).

The linearizing map is the involution

\[ (X, Z, A, B) \mapsto (Z^\top, X^\top, B^\top, A^\top). \]

All of that is old news. What we need to do now is show that there is a natural way to associate a bispectral matrix KP solution to each point of \( s\text{CM}^n_r \) (generalizing the known \( r = 1 \) case), and that the linearizing map corresponds to the bispectral involution.
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All of that is old news. What we need to do now is show that there is a natural way to associate a bispectrual matrix KP solution to each point of \( sCM_r^n \) (generalizing the known \( r = 1 \) case), and that the linearizing map corresponds to the bispectrual involution.
Bispectrality for Matrices...with a “twist”

- Not much has been done with bispectr-al matrix odos since Zubelli (1989). Seems unlikely that there are so many “unnoticed” bispectr-al matrix operators.
- His “bispectr-al problem” was in the form

\[
L\psi = \sum_{i=0} M_i(x) \frac{\partial^i}{\partial x^i} \psi(x, z) = p(z)\psi(x, z)
\]

\[
\Lambda\psi = \sum_{i=0} \hat{M}_i(z) \frac{\partial^i}{\partial z^i} \psi(x, z) = \pi(x)\psi(x, z).
\]

- This turns out not to be terribly rich. As we’ll see, the generalization of Wilson’s result requires us to look at

\[
\Lambda_R\psi = \sum_{i=0} \left( \frac{\partial^i}{\partial z^i} \psi(x, z) \right) \hat{M}_i(z) = \pi(x)\psi(x, z).
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- Not much has been done with bispectral matrix odos since Zubelli (1989). Seems unlikely that there are so many “unnoticed” bispectral matrix operators.
- His “bispectral problem” was in the form

\[ L\psi = \sum_{i=0} M_i(x) \frac{\partial^i}{\partial x^i} \psi(x, z) = p(z)\psi(x, z) \]

\[ \Lambda\psi = \sum_{i=0} \hat{M}_i(z) \frac{\partial^i}{\partial z^i} \psi(x, z) = \pi(x)\psi(x, z). \]

- This turns out not to be terribly rich. As we’ll see, the generalization of Wilson’s result requires us to look at

\[ \Lambda_R\psi = \sum_{i=0} \left( \frac{\partial^i}{\partial z^i} \psi(x, z) \right) \hat{M}_i(z) = \pi(x)\psi(x, z). \]
**Result 1: Analogue of Krichever Map**

**Definition:** To \((X, Z, A, B)\) associate the \(r \times r\) matrix odo \((x = t_1)\)

\[
W = \det(\partial I - Z)I + A(X - \sum it_iZ^{i-1})^{-1} adj(Z - \partial I)B.
\]

**Theorem:** \(\mathcal{L} = W \circ \partial \circ W^{-1}\) is a solution to the KP hierarchy.

**Key Steps of Proof:**

- Using matrix analysis and the rank \(r\) condition, we show that the kernel of \(W\) can be written nicely in terms of the residues of \(e^{\sum t_i z^i} / \det(zI - Z)\) at the eigenvalues of \(Z\).

- We then differentiate \(W\phi = 0\) wrt \(t_i\) and derive the "Lax equation"

\[
\dot{\mathcal{L}} = [\mathcal{L}, (\mathcal{L}^i)_+]\]

from it using differential algebra.

**Remark:** Wilson’s \(r = 1\) proof was similar, but was only handled \(Z\) with distinct eigenvalues. This proof “fills the hole”!
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Note that this is the matrix whose eigenvalue dynamics are governed by Spin Calogero...and hence so are the pole dynamics of the KP solution.
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**Result 2: Bispectrality**

**Definition:** For each choice of $(X, Z, A, B)$ and $\mathcal{L} = W \circ \partial \circ W^{-1}$ as before we define $L = p(\mathcal{L})$ where $p(z) = z^k \det(zI - Z)^2$ ($k \geq 0$).

**Theorem (Eigenfunction):** $L$ is an ordinary differential operator satisfying

$$L\psi = p(z)\psi \quad \text{for} \quad \psi(x, z) = e^{xz} \left( I + A(xI - X)^{-1}(zI - Z)^{-1}B \right).$$

Since this works for $k = 0$ and $k = 1$, we have commuting operators of relatively prime order.

**Definition:** Of course, we could do the same for $(Z^\top, X^\top, B^\top, A^\top)$ and get a different differential operator $L^b$ satisfying

$$L^b\psi^b(x, z) = \pi(z)\psi^b(x, z).$$

**Theorem (Bispectral Involution):** We use the simple fact that

$$\psi(x, z) = \left( \psi^b(z, x) \right)^\top$$

to conclude that $\Lambda = \left( L^b\big|_{x\to z} \right)^\top$ satisfies

$$\Lambda_R\psi(x, z) = \pi(x)\psi(x, z).$$
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- Having found this “coincidence” separately in numerous settings, I am inclined to think there must be some general theorem lurking behind the scenes.
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- Why does duality look like bispectrality for both quantum and classical systems? (Note: At the quantum level, the Hamiltonians are bispectral and classically it is individual states that are!)