Grassmannians, Nonlinear Wave Equations and Generalized Schur Functions

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ABSTRACT. A set of functions is introduced which generalizes the famous Schur polynomials and their connection to Grasmannian manifolds. These functions are shown to provide a new method of constructing solutions to the KP hierarchy of nonlinear partial differential equations. Specifically, just as the Schur polynomials are used to expand tau-functions as a sum, it is shown that it is natural to expand a *quotient* of tau-functions in terms of these generalized Schur functions. The coefficients in this expansion are found to be constrained by the Plücker relations of a grassmannian.

1. Introduction

The *KP hierarchy* of nonlinear partial differential equations is an important example of the interplay between dynamics and geometry. As dynamical systems, the equations of the KP hierarchy are used to model everything from ocean waves to elementary particles. However, the underlying mathematical structure is extremely geometric, being intimately related with the moduli of vector bundles over complex projective algebraic curves, coordinate systems for differential geometric objects, and especially the geometry of Grassmannian manifolds.

In this paper, a new set of functions is introduced which generalizes the well known Schur polynomials [**21, 24, 30**]. Like the Schur polynomials, these N-*Schur functions* are shown to be relevant to the KP hierarchy and directly related to its underlying geometric structure.

The remainder of this introduction contains a motivating example followed by a review of the theory of the KP hierarchy. Section 2 provides new definitions while Section 3 contains the new results. The paper concludes with a brief discussion in Section 4.

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1.1. An Example. Consider the nonlinear partial differential equation known as the *KP equation*:

$$\mathrm{KP}[u(x,y,t)] := \frac{3}{4}u_{yy} - \left(u_t - \frac{1}{4}(6uu_x + u_{xxx})\right)_x = 0 \tag{1.1}$$

This equation was originally derived in [**12**] as a model in fluid dynamics and contains as a one dimensional reduction (assuming $u_y = 0$) the famous KdV equation [**17**]. This equation shows up in many applications, for instance in describing ocean waves [**31**]. Moreover, although it is clearly an infinite dimensional dynamical system, it also contains as appropriate reductions certain classical finite dimensional Hamiltonian particle systems [**13, 18, 19, 27, 28, 33, 36**]. Forgetting about its many applications, this equation may not *appear* to be significantly different than other nonlinear PDEs. However, it actually is a remarkably special sort of equation in that it is *completely integrable*. This term has technical meaning, but here it will be used loosely to mean only that we can write many explicit solutions to this equation and even exactly solve the initial value problem for certain general classes of initial conditions (cf. [**8**]).

If the equation were *linear*, we would have a geometric interpretation for the set of solutions as a *vector space*. However, since (1.1) is nonlinear, we do not initially have any *geometric* notion of the solution space. The following example will demonstrate the sense in which solutions to the KP equation can similarly have the structure of an algebraic variety.

It is well known that one may construct solutions to the KP equation from Schur polynomials (cf. [**30**]). The main result of this paper is a generalization of this result. Without any explanation, let me provide you with six solutions constructed from the *generalized* Schur functions to be introduced below. Define

$$\tau_0 = \exp\left(\frac{-x^3}{6(1+3t)}\right) \qquad \tau_1 = 1$$

$$\begin{aligned}\tau_2 &= y + \frac{(1+3t)^{\frac{1}{3}}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{4^{\frac{4}{3}}}\left(-2^{\frac{1}{3}}x\left(3\mathrm{Ai}(\theta)^2 - \mathrm{Bi}(\theta)^2\right)\right.\\ &\quad \left. +(1+3t)^{\frac{1}{3}}\left(3\mathrm{Ai}'(\theta)^2 - \mathrm{Bi}'(\theta)^2\right)\right)\end{aligned}$$

$$\begin{aligned}\tau_3 &= \frac{(3+9t)^{\frac{1}{3}}\Gamma(\frac{2}{3})^2}{8}\left(-6x\mathrm{Ai}(\theta)^2 - 4\sqrt{3}x\mathrm{Ai}(\theta)\mathrm{Bi}(\theta)\right.\\ &\quad \left. -2x\mathrm{Bi}(\theta)^2 + 4^{\frac{1}{3}}(1+3t)^{\frac{1}{3}}\left(3\mathrm{Ai}'(\theta)^2 + 2\sqrt{3}\mathrm{Ai}'(\theta)\mathrm{Bi}'(\theta) + \mathrm{Bi}'(\theta)^2\right)\right)\end{aligned}$$

$$\begin{aligned}\tau_4 &= -\frac{1}{2} + \frac{(1+3t)^{\frac{1}{3}}\Gamma(\frac{1}{3})^2}{8(3^{\frac{5}{6}})}\left(-3(2^{\frac{1}{3}})\sqrt{3}x\mathrm{Ai}(\theta)^2 + 6(2^{\frac{1}{3}})x\mathrm{Ai}(\theta)\mathrm{Bi}(\theta)\right.\\ &\quad \left. -2^{\frac{1}{3}}\sqrt{3}x\mathrm{Bi}(\theta)^2 + (1+3t)^{\frac{1}{3}}\left(3\sqrt{3}\mathrm{Ai}'(\theta)^2 - 6\mathrm{Ai}'(\theta)\mathrm{Bi}'(\theta) + \sqrt{3}\mathrm{Bi}'(\theta)^2\right)\right)\end{aligned}$$

$$\begin{aligned}\tau_5 &= -y + \frac{(1+3t)^{\frac{1}{3}}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{4^{\frac{4}{3}}}\left(-2^{\frac{1}{3}}x\left(3\mathrm{Ai}(\theta)^2 - \mathrm{Bi}(\theta)^2\right)\right.\\ &\quad \left. +(1+3t)^{\frac{1}{3}}\left(3\mathrm{Ai}'(\theta)^2 - \mathrm{Bi}'(\theta)^2\right)\right)\end{aligned}$$

$$\tau_6 = \frac{-x(1+3t)}{2} + y^2 + \frac{(3+9t)^{\frac{1}{3}}\Gamma(\frac{2}{3})^2}{16}\Big(6x\mathrm{Ai}(\theta)^2 + 4\sqrt{3}x\mathrm{Ai}(\theta)\mathrm{Bi}(\theta)$$
$$+2x\mathrm{Bi}(\theta)^2 - 2^{\frac{2}{3}}(1+3t)^{\frac{1}{3}}\left(3\mathrm{Ai}'(\theta)^2 + 2\sqrt{3}\mathrm{Ai}'(\theta)\mathrm{Bi}'(\theta) + \mathrm{Bi}'(\theta)^2\right)\Big)$$

and

$$\theta = \frac{2^{\frac{1}{3}}x}{(1+3t)^{\frac{1}{3}}}$$

Here Ai and Bi are the standard Airy functions satisfying the differential equation $f''(x) = xf(x)$ and having Wronskian determinant $2/(\sqrt{3}\Gamma(1/3)\Gamma(2/3))$. Then note that each function

$$u_i(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log(\tau_0(x,t) * \tau_i(x,y,t)) \qquad (1 \le i \le 6)$$

is a solution to the KP equation[1]. The point here is not to understand *how* these solutions were found (this will be explained later), but rather to see the way in which they demonstrate the algebro-geometric structure underlying the equation.

Consider an arbitrary linear combination of these *tau-functions*:

$$\tau(x,y,t) := \tau_0 \cdot \sum_{i=1}^{6} \pi_i\tau_i \qquad \pi_i \in \mathbb{C}$$

and the corresponding function $u(x,y,t) = 2\partial^2/\partial x^2(\log\tau)$. We know that this function is a solution to (1.1) for *certain* choices of the coefficients π_i; in particular all but one could be equal to zero. *Do any other combinations lead to a KP solution?*

If every such combination gave a solution, then this would be a six-dimensional vector space of solutions. This is certainly not the case since, for instance, one *cannot* have all $\pi_i = 0$. Let us naively answer the question by simply inserting[2] this function $u(x,y,t)$ into Equation (1.1). One finds, after algebraic simplification, that $\mathrm{KP}[u] = 0$ is nothing but the homogeneous algebraic equation

$$\pi_1\pi_6 - \pi_3\pi_5 + \pi_2\pi_4 = 0.$$

This particular algebraic equation describes a well known projective geometric object: the Grassmannian $Gr_{2,4}$ [**11**].

The rest of this paper will attempt to explain this example, describe the way in which one can do the same to obtain the equations for any finite dimensional Grassmannian, and especially to emphasize its connection to the N-Schur functions.

1.2. The KP Hierarchy. To understand the structure underlying the integrability of (1.1), it is convenient to consider the KP equation as only part of an infinite hierarchy of equations, and to consider the variables x, y and t as only the first three variables in a hierarchy of infinitely many variables.

The KP hierarchy is an infinite set of compatible dynamical systems on the space of monic pseudo-differential operators of order one. A pseudo-differential

[1] That the function $u_1 = -2x/(3t+1)$ is a solution is simple enough to check by hand. I might suggest using a computer to verify the other solutions.

[2] Again, use a symbolic algebra computer program such as *Mathematica*. Actually, as the experts know, it is easier to use the bilinear form (cf. [**7**]):

$$\frac{3}{4}\tau\tau_{yy} - \frac{3}{4}\tau_y^2 + \tau_x\tau_t - \tau\tau_{xt} + \frac{3}{4}\tau_{xx}^2 - \tau_x\tau_{xxx} + \frac{1}{4}\tau\tau_{xxxx} = 0$$

of the KP equation rather than (1.1).

operator is a Laurent series in the symbol ∂ with coefficients that are functions of the variable x. The multiplication of these operators is defined by the relationships

$$\partial \circ f(x) = f(x)\partial + f'(x) \qquad \partial^{-1} \circ f(x) = f(x)\partial^{-1} - f'(x)\partial^{-2} + f''\partial^{-3} + \cdots$$

In other words, $\partial = d/dx$ and ∂^{-1} is its formal inverse. Contained within the ring of pseudo-differential operators is the ring of ordinary differential operators, those having only non-negative powers of ∂.

An *initial condition* for the KP hierarchy is any pseudo-differential operator of the form

$$\mathcal{L} = \partial + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \cdots . \tag{1.2}$$

The KP hierarchy is the set of dynamical systems defined by the evolution equations

$$\frac{\partial}{\partial t_i}\mathcal{L} = [\mathcal{L}, (\mathcal{L}^i)_+] \qquad i = 1, 2, 3, \ldots \tag{1.3}$$

where the "+" subscript indicates projection onto the differential operators by simply eliminating all negative powers of ∂ and $[A, B] = A \circ B - B \circ A$. In fact, since all of these flows commute (for $i = 1, 2, \ldots$) one can think of a solution of the KP hierarchy as a pseudo-differential operator of the form (1.2) whose coefficients depend on the time variables $t_1, t_2, \ldots$ so as to satisfy (1.3). (Note also that the first equation, $i = 1$, leads to the conclusion that $t_1 = x$ so these names will be used interchangeably. Similarly it is common to use $y = t_2$ and $t = t_3$.)

1.2.1. *The Tau-Function.* Remarkably, there exists a convenient way to encode all information about the KP solution $\mathcal{L}$ in a single function of the time variables $t_1, t_2, \ldots$. Specifically, each of the coefficients w_i of $\mathcal{L}$ can be written as a rational function of this function $\tau(t_1, t_2, \ldots)$ and its derivatives [**30**]. Alternatively, one can construct $\mathcal{L}$ from τ by letting W be the pseudo-differential operator

$$W = \frac{1}{\tau}\tau(t_1 - \partial^{-1}, t_2 - \frac{1}{2}\partial^{-2}, \ldots)$$

and then $\mathcal{L} := W \circ \partial \circ W^{-1}$ is a solution to the KP hierarchy [**1**].

Every solution to the KP hierarchy can be written this way in terms of a tau-function, though the choice of tau-function is not unique. For example, note that one may always multiply W on the right by any constant coefficient series $1 + O(\partial^{-1})$ without affecting the corresponding solution. More significant to the present paper is the elementary observation that multiplying the tau-function by any constant will not change the associated solution. So, it is reasonable to consider two tau-functions to be equivalent if they differ by multiplication of a constant (independent of $\{t_i\}$). Then, like the Plücker coordinates of a Grassmannian [**11**], the tau-function is uniquely defined only up to this *projective* equivalence.

1.3. Significance of the KP Hierarchy. If $\mathcal{L}$ is a solution to the KP hierarchy then the function

$$u(x, y, t) = -2\frac{\partial}{\partial x}w_1(x, y, t, \ldots) = 2\frac{\partial^2}{\partial x^2}\log\tau$$

is a solution of the KP equation (1.1). Moreover, many of the other equations that show up as particular reductions of the KP hierarchy have also been previously studied as physically relevant wave equations. The KP hierarchy also arises in *string theories* of quantum gravity [**22**], the probability distributions of the eigenvalues of random matrices [**2, 35**], and the description of coordinate systems in differential geometry [**9**].

Certainly one of the most significant comments which can be made regarding these equations, which is a consequence of the form (1.3), is that all of these equations are completely integrable. Among the many ways to solve the equations of the KP hierarchy are several with connections to the algebraic geometry of "spectral curves" [**5, 20, 25, 26, 30**]. (Conversely, through this same correspondence the KP hierarchy provides an answer to the famous Schottky problem in algebraic geometry [**23, 32**], representing *another* direction to the interaction between dynamics and geometry.) However, more relevant to the subject of this note is the observation of M. Sato that the geometry of an *infinite dimensional Grassmannian* underlies the solutions to the KP hierarchy [**29**].

1.3.1. *N-KdV and the Vector Baker-Akhiezer function.* Of particular interest below are the solutions $\mathcal{L}$ of the KP hierarchy that have the property that $L = \mathcal{L}^N = (\mathcal{L}^N)_+$ is an *ordinary* differential operator. We say that solutions of the KP hierarchy with this property are solutions of the of the N-KdV hierarchy. One can easily check from (1.3) that these solutions are stationary under all flows t_i where $i \equiv 0 \bmod N$. For instance, the solutions of the 2-KdV hierarchy are independent of all even indexed time parameters and consequently give solutions u to (1.1) which are independent of y and therefore solve the KdV equation.

Associated to any chosen solution $L = \mathcal{L}^N$ of the N-KdV hierarchy, we associate an N-vector valued function of the variables $\{t_i\}$ and the new *spectral parameter* z. Following [**25**] we define the *vector Baker-Akhiezer* function to be the unique function $\vec{\psi}(z, t_1, t_2, \dots)$ satisfying

$$L\vec{\psi} = z\vec{\psi} \qquad \frac{\partial}{\partial t_i}\vec{\psi} = (\mathcal{L}^i)_+\vec{\psi} \tag{1.4}$$

and such that the $N \times N$ Wronskian matrix

$$\Psi(z, t_1, t_2, \dots) := \begin{pmatrix} \vec{\psi} \\ \frac{\partial}{\partial x}\vec{\psi} \\ \vdots \\ \frac{\partial^{N-1}}{\partial x^{N-1}}\vec{\psi} \end{pmatrix} \tag{1.5}$$

is the identify matrix when evaluated at $0 = t_1 = t_2 = \dots$. (We will ignore here the case in which the matrix is undefined at this point. In fact, this is not a serious problem since such singularities are isolated and thus the problem can be resolved by using the KP flows.)

2. A Generalization of the Schur Functions

Let $N \in \mathbb{N}$ and consider the variables $h_k^{i,j}$ ($1 \le i, j \le N$, $k \in \mathbb{N}$) which can be conveniently grouped into $N \times N$ matrices

$$H_k = \begin{pmatrix} h_k^{1,1} & \cdots & h_k^{1,N} \\ \vdots & \ddots & \vdots \\ h_k^{N,1} & \cdots & h_k^{N,N} \end{pmatrix} \qquad k = 0, 1, 2, 3, 4, \dots$$

Moreover, these matrices will be grouped into the infinite matrix M_∞

$$M_\infty = \begin{pmatrix} \vdots & \vdots & \vdots & \cdots & & \\ H_2 & H_3 & H_4 & H_5 & \cdots & \cdots \\ H_1 & H_2 & H_3 & H_4 & \cdots & \cdots \\ H_0 & H_1 & H_2 & H_3 & \cdots & \cdots \\ 0 & H_0 & H_1 & H_2 & H_3 & \cdots \\ 0 & 0 & H_0 & H_1 & H_2 & \cdots \\ 0 & 0 & 0 & H_0 & H_1 & \cdots \\ \vdots & & & & \ddots & \end{pmatrix} \begin{matrix} \\ \\ \\ \}\text{rows } 0 \text{ through } N-1 \\ \}\text{rows } N \text{ through } 2N-1 \\ \\ \\ \\ \end{matrix}$$

It is convenient to label the rows of this matrix by the integers with 0 being the first row with H_0 at the left and increasing downwards.

We wish to define a set of functions in these variables indexed by partitions of integers. Specifically, the index set $\mathbb{S}$ will be the set of increasing sequences of integers whose values are eventually equal to their indices

$$\mathbb{S} = \{(s_0, s_1, s_2, \dots) \mid s_{j+1} > s_j \in \mathbb{Z} \text{ and } \exists m \text{ such that } j = s_j \ \forall j > m\}.$$

Of particular interest here will be the special case $0 := (0, 1, 2, 3, \dots) \in \mathbb{S}$.

Definition 2.1. For any $S \in \mathbb{S}$ let $f_S^N := \det(M_S \cdot M_0^{-1})$ where the infinite matrix M_S is the matrix whose j^{th} row is the s_j^{th} row of M_∞. Then we have, for instance, that

$$M_0 = \begin{pmatrix} H_0 & H_1 & H_2 & H_3 & \cdots & \cdots \\ 0 & H_0 & H_1 & H_2 & H_3 & \cdots \\ 0 & 0 & H_0 & H_1 & H_2 & \cdots \\ 0 & 0 & 0 & H_0 & H_1 & \cdots \\ \vdots & & & & \ddots & \end{pmatrix}.$$

Equivalently, one could say that the element in position (l, m) $(0 \le l, m \le \infty)$ of the matrix M_S is given by

$$(M_S)_{l,m} = h_k^{i,j} \qquad i = 1 + (l \bmod N),\ j = 1 + (s_m \bmod N),\ k = \left\lfloor \frac{l}{N} \right\rfloor - \left\lfloor \frac{s_m}{N} \right\rfloor$$

where $h_k^{i,j} = 0$ if $k < 0$.

Remark 2.2. The definition of M_S should remind one of the definition of the Plücker coordinates [**11**]. This is no coincidence; in fact we will see below that the N-Schur functions arise naturally in the context of an infinite dimensional Grassmannian.

Remark 2.3. You do not have to be comfortable with infinite matrices to work with the N-Schur functions. Note that given any $m \in \mathbb{N}$ such that $s_i = i$ for all $i > mN$, the matrix $M_S \cdot M_0^{-1}$ looks like the identity matrix below the mN^{th} row. Consequently, the easiest way to actually compute these functions is as two finite determinants

$$f_S^N = \frac{\det(M_S|_{mN \times mN})}{(\det H_0)^m}$$

where $M_S|_{mN\times mN}$ denotes the top left block of size $mN \times mN$ of the matrix M_S. For example, regardless of N, $f_0^N \equiv 1$. However, for $S = (-2, 1, 2, 3, \dots)$ one finds instead

$$f_S^1 = \frac{h_2^{1,1}}{h_0^{1,1}} \qquad f_S^2 = \frac{h_1^{1,1}h_0^{2,2} - h_0^{1,2}h_1^{2,1}}{h_0^{1,1}h_0^{2,2} - h_0^{1,2}h_0^{2,1}}.$$

REMARK 2.4. In fact, in the case $N = 1$ and $h_0^{1,1} = 1$, the functions $\{f_S^1\}$ are the famous Schur polynomials [**21**] (cf. [**24, 30**]). Similarly, if we assume in general that $\det H_0 = 1$, then all f_S^N are polynomials.

REMARK 2.5. If we consider $h_k^{i,j}$ to have weight $kN + i - j$ then the function f_S^N is homogeneous of weight $\sum_{j=0}^{\infty} s_j - j$.

3. Tau-Functions, Determinants on an Infinite Grassmannian and N-Schur Functions

Quotients of tau-functions have recently played a prominent role in several papers on bispectrality [**4, 16**], Darboux transformations [**3, 15**] and random matrices [**2**]. In [**15**] these quotients themselves are computed as a determinant of the action of a matrix valued function on the frame bundle of the grassmannian $Gr^N = Gr(H^N)$ (cf. [**25**]). Here we will see that such a determinant can be decomposed into a sum of N-Schur functions when given appropriate dependence on the time variables of the KP hierarchy.

3.1. The Grassmannian Gr^N. Let us recall notation and some basic facts about infinite dimensional grassmannians. Please refer to [**15, 24, 25, 30**] for additional details.

Let $H^N = L^2(S^1, \mathbb{C})$ be the Hilbert space of square-integrable vector valued functions $S^1 \to \mathbb{C}^N$, where $S^1 \subset \mathbb{C}$ is the unit circle. Denote by e_i $(0 \le i \le N-1)$ the N-vector which has the value 1 in the $i+1^{st}$ component and zero in the others. We fix as a basis for H^N the set $\{e_i | i \in \mathbb{Z}\}$ with

$$e_i := z^{\lfloor \frac{i}{n} \rfloor} e_{(i \bmod N)}.$$

The Hilbert space has the decomposition

$$H^N = H_+^N \oplus H_-^N \tag{3.1}$$

where these subspaces are spanned by the basis elements with non-negative and negative indices respectively. Then Gr^N denotes the grassmannian of all closed subspaces $W \subset H^N$ such that the orthogonal projection $W \to H_-^N$ is a compact operator and such that the orthogonal projection $W \to H_+^N$ is Fredholm of index zero [**24, 25**].

Associate to any basis $\{w_0, w_1, \dots\}$ for a point $W \in Gr^N$ the linear map w

$$\begin{array}{rcl} w : H_+^N & \to & W \\ e_i & \mapsto & w_i. \end{array}$$

The basis is said to be *admissible* if w differs from the identity by an element of trace class [**34**]. The *frame bundle* of Gr^N is the set of pairs (W, w) where $W \in Gr^N$ and $w : H_+^N \to W$ is an admissible basis.

There is a convenient way to embed Gr^N in a projective space. Let Λ denote the infinite alternating exterior algebra generated by the alternating tensors

$$\{e_{s_0} \wedge e_{s_1} \wedge e_{s_2} \wedge \cdots | (s_0, s_1, s_2, \dots) \in \mathbb{S}\}.$$

To any point (W,w) in the frame bundle we associate the alternating tensor

$$|w\rangle := w_0 \wedge w_1 \wedge w_2 \wedge \cdots \in \Lambda.$$

Note in particular that $|\cdot\rangle$ is projectively well defined on the entire fiber of W (i.e. for two admissible bases of W we have $|w\rangle = \lambda|w'\rangle$ for some non-zero constant λ). Consequently, $|W\rangle$ is a well defined element of the projective space $\mathbb{P}\Lambda$.

The Plücker coordinates of W are the coefficients $\langle S|W\rangle$ in the unique expansion

$$|W\rangle = \sum_{S\in\mathbb{S}} \langle S|W\rangle e_{s_0} \wedge e_{s_1} \wedge e_{s_2} \wedge \cdots$$

and are therefore well defined as a set up to a common multiple. Alternatively, given an admissible basis w for W, $\langle S|W\rangle$ is the determinant of the infinite matrix made of the rows of w indexed by the elements of S.

3.2. Main Results. Let g be an $N \times N$ matrix valued function of z with expansion

$$g = \sum_{k=0}^{\infty} H_k z^k \qquad H_k = \begin{pmatrix} h_k^{1,1} & \cdots & h_k^{1,N} \\ \vdots & \ddots & \vdots \\ h_k^{N,1} & \cdots & h_k^{N,N} \end{pmatrix} \tag{3.2}$$

such that an inverse matrix g^{-1} exists for all z. We will view $g \in GL(H^N)$ as an operator on Gr^N and demonstrate that the N-Schur functions arise naturally in this context.

In general, an operator on the frame bundle **[24]** is a pair $A = (g,q)$ where $g \in GL(H^N)$ with the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.3}$$

relative to the splitting (3.1) and $q : H_+^N \to H_+^N$ such that $a \cdot q^{-1}$ differs from the identity by an operator of trace class. The action is given by

$$A : (W,w) \mapsto (gW, gwq^{-1}).$$

In the particular case (3.2) of interest here $c = 0$ and we simply let $q = a$ so that aq^{-1} is the identity matrix. We will write $|g|w\rangle = |gwa^{-1}\rangle$ for the action of g on the frame bundle. Moreover, since this action is well defined on projective equivalence classes we will write $|g|W\rangle$ for the class containing $|g|w\rangle$ with any admissible basis w of W.

A main result of this paper is then the observation that for any point $W \in Gr^N$ the determinant $\langle 0|g|W\rangle$ can be written in terms of the Plücker coordinates of W and the N-Schur functions:

THEOREM 3.1. *For $W \in Gr^N$ and g as in* (3.2)

$$\langle 0|g|W\rangle = \sum_{S\in\mathbb{S}} \langle S|W\rangle f_S^N.$$

PROOF. The proof is elementary in the case $W = W_S$ with the usual basis $\{e_{s_0}, e_{s_1}, e_{s_2}, \dots\}$. In fact, this is essentially the definition of f_S^N since the matrix representation of the operator g is precisely M_∞ and similarly q is M_0. So, gwq^{-1} is the matrix $M_S M_0^{-1}$ whose determinant is f_S^N. The general case follows from the observation that multilinearity of determinants is equivalent to the linearity of the map $\langle 0|g| : \Lambda \to \mathbb{C}$ and expanding $|W\rangle$ as a sum. □

This general result is especially interesting in the case that the variables $h_k^{i,j}$ are evaluated as special functions of the KP times t_i associated to the choice of a solution of the N-KdV hierarchy. Specifically, associated to the choice of a solution $\mathcal{L}$ of the N-KdV hierarchy define

$$h_k^{i,j} := \frac{1}{k!}\frac{\partial^k}{\partial z^k}(\Psi^{-1})_{ij}\bigg|_{z=0} \tag{3.4}$$

where Ψ is the corresponding Wronskian matrix. In that case each N-Schur function is a quotient of KP tau-functions:

THEOREM 3.2. *Let $\mathcal{L}$ be a solution of the N-KdV hierarchy with corresponding tau-function τ_0 and corresponding matrix Ψ given in* (1.4) *and* (1.5)*. Give the N-Schur functions dependence on the time variables of the KP hierarchy through* (3.4) *so that*

$$\Psi^{-1} = \sum_{k=0}^{\infty} H_k z^k.$$

Then there exists a tau-function τ_S of the KP hierarchy so that

$$f_S^N(t_1, t_2, \dots) = \frac{\tau_S}{\tau_0}$$

for every $S \in \mathbb{S}$. Moreover, it follows that

$$\tau_0 \cdot (\sum_{S\in\mathbb{S}} \pi_S f_S^N)$$

is a tau-function of the KP hierarchy whenever π_S are the Plücker coordinates of some point in Gr^N.

PROOF. Defining $g = \Psi^{-1} \in GL(H^n)$ where Ψ is the matrix (1.5) above, the determinant $\langle 0|g|W\rangle$ is a (projective) function of the variables t_i. It is shown in [**15**] (Definition 7.4 and Claim 7.12) that these functions are quotients of KP tau-functions with a tau-function corresponding to $\mathcal{L}$ in the denominator. Consequently, using the theorem above we may write these quotients in terms of the N-Schur functions to prove the claim. □

This is, of course, a generalization of the well known result relating Schur polynomials and the KP hierarchy. In particular, that result is the special case of the 1-KdV solution $L = \partial$ for which

$$\Psi = \exp(\sum t_i z^i).$$

In that case, of course, the time dependent polynomials in the variables $h_k^{i,j}$ are also polynomial in the variables t_i. In general, that will not be the case.

3.3. Finite Dimensional Grassmannians. This would then be a good time to describe the construction of the solutions to the KP equation given in Section 1.1. Since that example concerned only the KP equation (and not all of the equations of the hierarchy) we need only consider the first three time variables $t_1 = x$, $t_2 = y$

and $t_3 = t$. One well known but surprisingly complicated[3] solution to the 2-KdV hierarchy is $L_0 = \mathcal{L}^2 = \partial^2 - 2x/(3t+1)$. It corresponds to the tau-function τ_0 in the example. The other functions τ_i are just 2-Schur functions given time dependence by

$$\begin{aligned} \Psi^{-1} &= \phi(y,z) * \begin{pmatrix} \frac{-\mathrm{Ai}'(\Theta)\mathrm{Bi}(\zeta)+\mathrm{Ai}(\zeta)\mathrm{Bi}'(\Theta)}{2(1+3t)^{\frac{1}{6}}} & \frac{(1+3t)^{\frac{1}{6}}(-\mathrm{Ai}(\zeta)\mathrm{Bi}(\Theta)+\mathrm{Ai}(\Theta)\mathrm{Bi}(\zeta))}{22^{\frac{1}{3}}} \\ \frac{\mathrm{Ai}'(\zeta)\mathrm{Bi}'(\Theta)-\mathrm{Ai}'(\Theta)\mathrm{Bi}'(\zeta)}{2^{\frac{2}{3}}(1+3t)^{\frac{1}{6}}} & \frac{-(1+3t)^{\frac{1}{6}}\left(\mathrm{Ai}'(\zeta)\mathrm{Bi}(\Theta)-\mathrm{Ai}(\Theta)\mathrm{Bi}'(\zeta)\right)}{2} \end{pmatrix} \\ &= \sum_{k=0}^{\infty} H_k z^k \end{aligned}$$

with $\phi(y,z) = \sqrt{3}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})e^{-yz}$, $\zeta = 4^{-\frac{1}{3}}z$ and $\Theta = \frac{3tz+2x+z}{2^{\frac{2}{3}}(1+3t)^{\frac{1}{3}}}$. Then considering Theorem 3.2 with the additional restriction that all but these six coordinates must be zero gives exactly the Plücker relation for $Gr_{2,4}$. More generally:

DEFINITION 3.3. Let $k < n \in \mathbb{N}$ be positive integers and define $\mathbb{S}_{k,n} \subset \mathbb{S}$ as

$$\mathbb{S}_{k,n} = \{S \in \mathbb{S} \mid k-n \le s_i \le k-1 \ (0 \le i \le k-1)\}.$$

Note that $\mathbb{S}_{k,n}$ contains exactly $\binom{n}{k}$ elements. Specifically, every element of $\mathbb{S}_{k,n}$ corresponds to a choice of k integers between $k-n$ and $k-1$. Let

$$\gamma_{k,n}(t_1, t_2, \dots) := \sum_{S \in \mathbb{S}_{k,n}} \pi_S f_S^N(t_1, t_2, \dots)$$

where $\pi_S \in \mathbb{C}$ are arbitrary parameters and define $\tau_{k,n} := \tau_0 \cdot \gamma_{k,n}$.

We are naturally led to consider the coefficients $\{\pi_S\}$ as points in the projective space $\mathbb{P}^{m-1}\mathbb{C}$ ($m = \binom{n}{k}$) and to ask: *For what points in this projective space is $\tau_{k,n}$ a tau-function?* The answer provided by Theorem 3.2 above is simply:

COROLLARY 3.4. *The function $\tau_{k,n}(t_1, t_2, \dots)$ depending on the $\binom{n}{k}$ parameters π_S is a KP tau function precisely when they satisfy the Plücker relations for the Grassmannian $Gr_{k,n}$.*

So, this gives us a procedure for deriving the algebraic Plücker relations from the differential equations of the KP hierarchy as in the example.

4. Discussion

That the (1-)Schur polynomials give solutions to the KP hierarchy is often cited as having been the *clue* which led Sato to recognize the connection between KP and Grassmannians [**29**]. This well known case is especially nice since it comes from N-Schur functions associated to the simplest solution $\tau = 1$ and it is used to expand the tau-function as a sum. However, the more general situation discussed here has similar applications in expanding *quotients* of tau-functions. (The Schur polynomials correspond to the special case in which the tau-function in the

[3] I say that this solution is surprisingly complicated because it does not come from any of the usual methods of solution. This solution is *not* related to a flow on a Jacobian variety of a spectral curve, since it is "rank 2". This solution is not solvable by the inverse scattering method since it certainly does not vanish for $x \to \infty$. This solution is not even among the many analytically determined solutions in [**30**]. Moreover, it is this solution which was related to intersection numbers on an algebro-geometric moduli space by a conjecture of Witten and theorem of Kontsevich leading to a Fields' Medal for the latter.

denominator is equal to 1, so the tau-quotient is itself also a tau-function.) Such tau-quotients associated with Darboux transformations are themselves of interest **[2, 15]** and the introduction of the N-Schur functions provides a means to expand these as sums with coefficients constrained by algebraic equations.

One thing which is not yet clear, at least to me, is whether the N-Schur functions have any group theoretic significance in the case $N > 1$. Certainly, in the case $N = 1$ there are very satisfying explanations of the role of the Schur polynomials in the KP Hierarchy in terms of their orthogonality **[29]**, highest-weight conditions **[7, 10]** or representations of Heisenberg algebras **[6]**. It would be interesting if one could generalize these algebraic interpretations of the tau-functions of the KP hierarchy to the situation discussed above.

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