

\mathcal{D} -MODULES AND DARBOUX TRANSFORMATIONS

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ABSTRACT. A method of G. Wilson for generating commutative algebras of ordinary differential operators is extended to higher dimensions. Our construction, based on the theory of \mathcal{D} -modules, leads to a new class of examples of *commutative rings of partial differential operators* with rational spectral varieties. As an application, we briefly discuss their link to the bispectral problem and to the theory of lacunas.

1. INTRODUCTION

1.1. The purpose of this paper is to discuss a certain algebro-geometric construction of commutative rings of partial differential operators with singular spectral varieties. The principal idea of the method to be used goes back to the work of Krichever (see, e.g., [25] for a review), though our effort here has been inspired by more recent developments. These are the study of algebraic Schrödinger operators initiated by O. Chalykh and A. Veselov [15, 16] and the work of G. Wilson [37, 38] on “bispectral” algebras of ordinary differential operators.

Our attempt is to merge the ideas of the above mentioned authors within a more general setting and to explore some new instructive examples.

1.2. In [15]–[16] a certain geometric axiomatics for the Baker-Akhiezer (BA) function has been developed in affine spaces of arbitrary dimension ($n \geq 1$). The construction involves a finite set \mathcal{A} of linear (homogeneous) hyperplanes in \mathbb{C}^n with prescribed (integer) multiplicities. These data are postulated to define a “pole divisor” of an associated BA function. When it exists, such a function is necessarily unique and solves a (formal) spectral problem for a certain *commutative* ring $\mathcal{R}_{\mathcal{A}}$ of partial differential operators containing a second order (Schrödinger) operator. Algebraically, the ring $\mathcal{R}_{\mathcal{A}}$ turns out to be *supercomplete* in the sense that the minimal number of generators of $\mathcal{R}_{\mathcal{A}}$ (as a \mathbb{C} -algebra) exceeds its Krull dimension, i.e. the dimension of the spectrum $\text{Spec } \mathcal{R}_{\mathcal{A}}$, while geometrically the latter proves to be a singular variety. Within the “rational” version of this axiomatics, the differential operators in $\mathcal{R}_{\mathcal{A}}$ have *constant* principal symbols and rational lower order coefficients with singularities located on the hyperplanes of \mathcal{A} . These operators possess a number of interesting analytic properties and applications among which one should mention the link to Huygens’ principle and the theory of lacunas (see [5, 6, 7, 8]).

The principal problem within the Veselov-Chalykh approach is to determine all possible hyperplane arrangements for which a nontrivial BA function does actually exist. Up to now, this problem has been completely settled in dimension $n = 2$ (see [3, 4]), while in higher dimensions ($n \geq 3$) only partial results and examples are available (see [33, 34, 35]).

1.3. One may pose a more general question (cf. [25], Section 4): What algebraic (or analytic) sets in \mathbb{C}^n are admissible as a singularity locus for the BA function

associated to a (non-trivial) supercomplete commutative ring of partial differential operators?

In this paper we give a partial answer to this question in the “rational” (algebraic) situation. Let X be any open quasi-affine algebraic variety in \mathbb{C}^n , i.e. $X = \mathbb{C}^n \setminus \tau^{-1}(0)$ with some $\tau \in \mathbb{C}[x_1, \dots, x_n]$, and let $\mathcal{O}(X)$ and $\mathcal{D}(X)$ stand for the rings of regular functions and differential operators on X respectively. It is a consequence of Theorem 4.2 below that there exists a nontrivial supercomplete commutative ring $\mathcal{R}_X \subset \mathcal{D}(X)$ of differential operators on X whose common eigenfunction $\psi = \psi(x, \xi)$ is unique (up to a normalization) and quasi-regular, i.e. has the form $\psi(x, \xi) := P(x, \xi)e^{(x, \xi)}$ with $P \in \mathcal{O}(T^*X) \cong \mathcal{O}(X) \otimes \mathbb{C}[\xi]$.

This result implies that the BA function (regarded as a function of x only) can admit singularities along *arbitrary* algebraic hypersurfaces in \mathbb{C}^n . In Section 4 we will give a simple and constructive proof of this observation. A number of explicit examples will be treated in Section 5.

It is worth mentioning that the ring \mathcal{R}_X does not necessarily contain a second order operator and, hence, our examples go beyond the framework of the Veselov-Chalykh axiomatics; however, as we will see in Section 5, they have equally interesting analytic applications.

Unlike the BA functions that arise from the Veselov-Chalykh construction, a generic ψ -function associated with a ring \mathcal{R}_X is *not self-dual*. Nevertheless, we demonstrate the existence of a complete ring of partial differential operators in the *spectral parameter* having ψ as a common eigenfunction. In this way, we get an example of a nontrivial bispectral involution in the case of several variables.

1.4. For ordinary differential operators, a *complete* answer to the question stated above is a simple consequence of a method of G. Wilson [37]. His technique was reformulated later in terms of Darboux transformations in [1, 2] and [21, 22]. The method used in the present work can be regarded as a further generalization of this approach with a view to address the problem for *partial* differential operators ($n > 1$). A natural framework for such a generalization is provided by *the theory of \mathcal{D} -modules*. Our intention is to demonstrate that the language of \mathcal{D} -modules is quite relevant and illuminating in this context. It brings some new insight even in the case $n = 1$ (see Section 3). In this paper we focus mostly on the construction of families of commuting differential operators rather than on the study of their properties. In particular, we only briefly discuss the bispectral problem which was one of our original motivations. We believe, however, that the understanding of this phenomenon within the theory of \mathcal{D} -modules is a promising problem which deserves a separate and more detailed investigation.

2. THE THEORY OF \mathcal{D} -MODULES: BASIC CONCEPTS

We start with a brief summary of some results and definitions from the theory of \mathcal{D} -modules. Our consideration will be restricted to the purely global (algebraic) case. For a more general and comprehensive treatment of the subject, the reader is referred to the literature [12, 10, 27, 28] (see also [29, 30]).

2.1. Regular Differential Operators. Let X be a non-singular irreducible affine algebraic variety over \mathbb{C} . We may (and will) regard X as being imbedded in the standard n -dimensional affine space \mathbb{C}^n , i.e. we identify X with a set of common zeros of a (prime) ideal $\mathcal{I}(X)$ in the polynomial ring $\mathbb{C}[x]$. Let $\mathcal{O}(X)$ be an *affine*

algebra associated with X (the coordinate ring of *regular* functions on X). By our identification, we have $\mathcal{O}(X) \cong \mathbb{C}[x]/\mathcal{I}(X)$. Then, $\mathcal{O}(X)$ is an integral domain. We write $\mathcal{F}(X)$ for the quotient field of $\mathcal{O}(X)$.

Let $\mathfrak{g}_{\mathcal{F}(X)}$ be the set of all \mathbb{C} -linear derivations on $\mathcal{F}(X)$ and let $\mathfrak{g}_{\mathcal{O}(X)} \subset \mathfrak{g}_{\mathcal{F}(X)}$ be its proper subset that preserves the subring $\mathcal{O}(X) \subset \mathcal{F}(X)$, i.e. $\mathfrak{g}_{\mathcal{O}(X)} := \{\partial \in \mathfrak{g}_{\mathcal{F}(X)} \mid \partial \in \text{End}_{\mathbb{C}}(\mathcal{O}(X))\}$.

Definition 2.1. The *ring of regular (algebraic) differential operators* on X is the ring of \mathbb{C} -linear endomorphisms of $\mathcal{O}(X)$ generated by the derivation operators from $\mathfrak{g}_{\mathcal{O}(X)}$ and the multiplication operators from $\mathcal{O}(X)$ itself. This ring is denoted by $\mathcal{D}(X)$.

The following examples will be of basic concern in the present work.

Example 1. If $X = \mathbb{C}^n$ then $\mathcal{D}(X)$ is the *Weyl algebra* $A_n(\mathbb{C}) \cong \mathbb{C}\langle x, \frac{\partial}{\partial x} \rangle$, i.e. the algebra of differential operators with polynomial coefficients.

Example 2. Let X be any smooth algebraic variety. For $\tau \in \mathcal{O}(X)$, $1/\tau$ is defined as a regular function on the complement $X_\tau := X \setminus \tau^{-1}(0)$ of the zero set of τ in X . We write $\mathcal{O}(X)_\tau$ for the algebra generated by $\mathcal{O}(X)$ and $1/\tau$ and call it the *localization* of the ring $\mathcal{O}(X)$ at τ . Similarly, the ring $\mathcal{D}(X_\tau)$ of regular differential operators on X_τ can be identified with a *localization* of $\mathcal{D}(X)$ at τ , i.e.

$$(1) \quad \mathcal{D}(X_\tau) \cong \mathcal{D}(X)_\tau := \mathcal{O}(X)_\tau \otimes_{\mathcal{O}(X)} \mathcal{D}(X).$$

In particular, if $X = \mathbb{C}^n$ then $X_\tau = \mathbb{C}^n \setminus \tau^{-1}(0)$, where $\tau \in \mathbb{C}[x]$, is an open (quasi-)affine variety of dimension n and the ring $\mathcal{D}(X_\tau)$ is identified with the *localized Weyl algebra* $A_n[\tau^{-1}] := \mathbb{C}[x, \tau^{-1}] \otimes_{\mathbb{C}[x]} A_n(\mathbb{C})$.

2.2. \mathcal{D} -modules. In general, the ring $\mathcal{D}(X)$ carries a natural filtration $\mathcal{O}(X) \equiv \mathcal{D}_0(X) \subset \mathcal{D}_1(X) \subset \mathcal{D}_2(X) \subset \dots$ provided by orders of differential operators (so that \mathcal{D}_k is the subspace of operators of order $\leq k$) with $[\mathcal{D}_k, \mathcal{D}_l] \subset \mathcal{D}_{k+l-1}$ and $\mathcal{D}(X) = \cup_{k \geq 0} \mathcal{D}_k(X)$. The *associated graded ring* $\text{gr}(\mathcal{D}(X)) := \oplus_{k \geq 0} \mathcal{D}_k/\mathcal{D}_{k-1}$ is a commutative noetherian ring isomorphic to the affine algebra of the cotangent bundle of X , i.e. $\text{gr}(\mathcal{D}(X)) \cong \mathcal{O}(T^*X)$. There is a natural mapping $\sigma : \mathcal{D}(X) \rightarrow \text{gr}(\mathcal{D}(X))$ which associates to a differential operator $Q \in \mathcal{D}(X)$ its *principal symbol* $\sigma(Q)$, a polynomial function on T^*X homogeneous along the fibers of T^*X . The following properties are immediate: (a) $\sigma(Q_1 Q_2) = \sigma(Q_1) \sigma(Q_2)$; (b) if $\mathfrak{A} \subset \mathcal{D}(X)$ is a left ideal in $\mathcal{D}(X)$ then $\sigma(\mathfrak{A})$ is an ideal in $\text{gr}(\mathcal{D}(X))$; moreover, (c) if $Q_1, \dots, Q_l \in \mathfrak{A}$ can be chosen in such a way that $\sigma(Q_1), \dots, \sigma(Q_l)$ generate $\sigma(\mathfrak{A})$ in $\text{gr}(\mathcal{D}(X))$ then Q_1, \dots, Q_l generate \mathfrak{A} in $\mathcal{D}(X)$. As a consequence, we note that $\mathcal{D}(X)$ is a left (right) noetherian ring.

Let M be a left unitary $\mathcal{D}(X)$ -module. A *\mathcal{D} -filtration* on M is a sequence $M_0 \subseteq M_1 \subseteq \dots$ of $\mathcal{O}(X)$ -submodules such that (i) $\cup_{k \geq 0} M_k = M$ and (ii) $\mathcal{D}_k M_l \subseteq M_{k+l}$ for all $k, l \in \mathbb{Z}_+$. If the last inclusion (ii) holds with an equality for all sufficiently large $l \in \mathbb{Z}_+$ then the \mathcal{D} -filtration is said to be *good*. In fact, M can be equipped with a good \mathcal{D} -filtration if and only if it is finite over $\mathcal{D}(X)$. Given a system of generators $\{e_1, \dots, e_s\}$ of a finite \mathcal{D} -module, one can define a good \mathcal{D} -filtration by setting $M_k := \mathcal{D}_k \langle e_1, \dots, e_s \rangle$ for every $k \in \mathbb{Z}_+$. Such \mathcal{D} -filtrations are called *standard*, any two of them, say $\{M_k\}$ and $\{M'_k\}$, being equivalent in the sense that $M_k \subseteq M'_{k+k_0}$ and $M'_k \subseteq M_{k+k_0}$ for some $k_0 \in \mathbb{Z}_+$ and for all $k \in \mathbb{Z}_+$.

The important geometric invariant of a \mathcal{D} -module M is its *characteristic variety* V_M . Let, for simplicity, $X = \mathbb{C}^n$, and let M be a finite module on X over the Weyl

algebra $\mathcal{D} = A_n(\mathbb{C})$. Consider the graded $\text{gr}(\mathcal{D})$ -module $\text{gr}_{\mathcal{D}}(M) := \bigoplus_{k \geq 0} M_k/M_{k-1}$ associated with some good \mathcal{D} -filtration on M and take its left annihilating ideal in $\text{gr}(\mathcal{D})$:

$$(2) \quad I_{\mathcal{D}}(M) := \{\sigma \in \text{gr}(\mathcal{D}) : \sigma \cdot \text{gr}_{\mathcal{D}}(M) = 0\}.$$

The variety V_M is then defined as the zero set of $I_{\mathcal{D}}(M)$ in $T^*X \cong \mathbb{C}^{2n}$, i.e.

$$(3) \quad V_M := \{(x, \xi) \in \mathbb{C}^{2n} : \sigma(x, \xi) = 0 \text{ for all } \sigma \in I_{\mathcal{D}}(M)\}.$$

Clearly, V_M is a ξ -conic algebraic variety in \mathbb{C}^{2n} whose “vanishing” ideal \mathcal{J}_M is equal to the radical of $I_{\mathcal{D}}(M)$ (Hilbert’s Nullstellensatz) and called the *characteristic ideal* of M . It turns out that both V_M and \mathcal{J}_M depend only on M and not on the filtration chosen.

If $\pi : T^*X \rightarrow X$ is the canonical projection, one defines the *support* of a \mathcal{D} -module M as a π -image of its characteristic variety

$$(4) \quad \text{supp}(M) := \pi(V_M) = \{x \in \mathbb{C}^n : \exists(x, \xi) \in V_M\}.$$

The dimension of V_M is called the *Bernstein dimension* $d(M)$ of M . It is a fundamental fact in the theory (cf. [10, 11]) that $d(M) \geq n$ for every nonzero M . This suggests that one should single out the class of so-called *holonomic* \mathcal{D} -modules \mathcal{B}_n (the *Bernstein class*) for which $d(M) = n$. Holonomic \mathcal{D} -modules possess a number of finiteness properties that make them of particular interest from the analytic point of view (see, e.g., [12]). For example, every $M \in \mathcal{B}_n$ has finite length (i.e., any descending chain of \mathcal{D} -submodules in M becomes stationary). In view of Stafford’s theorem, this implies the cyclicity of M since \mathcal{D} is a simple ring of infinite length. As a whole, the Bernstein class \mathcal{B}_n is stable under certain *basic operations* on \mathcal{D} -modules such as taking direct-inverse images, localization and the Fourier transform.

In general, the precise determination of a characteristic variety is a difficult problem, even in the case of holonomic \mathcal{D} -modules. On the other hand, if a \mathcal{D} -module M has a characteristic variety V_M of a particularly simple structure one can expect to describe it explicitly. In Section 3, we illustrate this remark in the *simplest* case: for holonomic \mathcal{D} -modules $M \in \mathcal{B}_1$ supported at a finite set of points in the complex plane, $\text{supp}(M) = \{\lambda_1, \dots, \lambda_s\}$. Using the basic operations on \mathcal{D} -modules we then recover the Wilson construction of rank one commutative algebras of ordinary differential operators. The very existence of such algebras reflects a certain symmetry (or, to be more precise, “simplicity”) of the underlying \mathcal{D} -module and its characteristic variety.

3. HOLONOMIC \mathcal{D} -MODULES AND COMMUTATIVE ALGEBRAS OF ORDINARY DIFFERENTIAL OPERATORS

In this section we review a construction of rank one commutative rings of differential operators in dimension $n = 1$ due to G. Wilson [37]. Our purpose is to analyze Wilson’s scheme in terms of holonomic \mathcal{D} -modules over the Weyl algebra and its localizations. Reinterpreting Wilson’s construction in these terms motivates the higher dimensional generalization to be presented in Section 4.

3.1. \mathcal{D} -modules with point support. Let $X = \mathbb{C}^n$ and let M be a finite left \mathcal{D} -module on X with $\mathcal{D} \cong A_n(\mathbb{C})$. Suppose that $\text{supp}(M) = \{\lambda\}$, $\lambda \in \mathbb{C}^n$. In this case one can recover the structure of M explicitly. Indeed, it is clear that $\dim_{\mathbb{C}} V_M \leq n$, where V_M is the characteristic variety of M . In view of Bernstein’s inequality this

implies $d(M) = n$ and $V_M = T_\lambda^* \mathbb{C}^n$. Hence, M is a holonomic \mathcal{D} -module, $M \in \mathcal{B}_n$. If $M_0 \subset M_1 \subset M_2 \subset \dots$ is a good filtration on M , the associated graded module $\text{gr}_{\mathcal{D}} M$ is supported at λ . By the Nullstellensatz one has

$$(5) \quad \mathfrak{m}_\lambda^N \sigma(a) = 0$$

with some $N \in \mathbb{Z}_+$ for any $\sigma(a) \in \text{gr}_{\mathcal{D}}(M)$, where $\mathfrak{m}_\lambda \equiv \mathbb{C}[x]\langle x - \lambda \rangle$ is the (maximal) ideal in $\mathbb{C}[x]$ corresponding to the point λ . But then, (5) is also true for any element in M : $\mathfrak{m}_\lambda^N a = 0$. To clarify the structure of M we start with the simplest example: $\Delta(\lambda) := \mathcal{D}/\mathcal{D}\mathfrak{m}_\lambda$. Clearly, $\text{supp } \Delta(\lambda) = \{\lambda\}$ since $\mathfrak{m}_\lambda^{N+1} Q \in \Delta(\lambda)$ for every $Q \in \mathcal{D}$ of order $\text{ord } Q \leq N$. Denote by $\delta_\lambda(x)$ the canonical generator of $\Delta(\lambda)$: $\delta_\lambda(x) := 1 + \mathcal{D}\mathfrak{m}_\lambda$ (so $\delta_\lambda(x)$ is the image of $1 \in \mathcal{D}$ under the natural surjection $\mathcal{D} \rightarrow \Delta(\lambda)$). It is easy to verify that

$$\Delta(\lambda) \cong \mathcal{D}\langle \delta_\lambda(x) \rangle \cong \mathbb{C}[\partial]\langle \delta_\lambda(x) \rangle,$$

where $\mathbb{C}[\partial]$ is the ring of differential operators with constant coefficients in \mathbb{C}^n .

The following lemma determines the algebraic structure of a finite \mathcal{D} -module with point support.

Lemma 3.1. *If M is a finite-type left \mathcal{D} -module on $X = \mathbb{C}^n$ with $\text{supp } M = \{\lambda\}$, $\lambda \in \mathbb{C}^n$, then M is isomorphic to a finite direct sum of copies of $\Delta(\lambda) = \mathcal{D}/\mathcal{D}\mathfrak{m}_\lambda$. In particular, $\Delta(\lambda)$ is a simple \mathcal{D} -module (i.e. has no nontrivial submodules over \mathcal{D}).*

For a proof of this result the reader is referred to [12], Ch. 5 (see also [29, 30]).

3.2. Left Ideals of the Weyl Algebra. Now let $n = 1$ and let M be any finite left \mathcal{D} -module over the Weyl algebra $\mathcal{D} = A_1(\mathbb{C})$. We choose a system of generators $\{e_1, \dots, e_s\}$, $M = \mathcal{D}\langle e_1, \dots, e_s \rangle$, and fix the standard filtration $M_1 \subset M_2 \subset \dots$ on M associated with this system $M_k := \mathcal{D}_k\langle e_1, \dots, e_s \rangle$. Then, we denote by $\mathfrak{A} = \mathfrak{A}(e_1, \dots, e_s) \subset \mathcal{D}$ the left ideal in \mathcal{D} that annihilates the elements $\{e_1, \dots, e_s\}$

$$(6) \quad \mathfrak{A} := \text{Ann}_{\mathcal{D}}(e_1, \dots, e_s) = \{Q(x, \partial) \in \mathcal{D} : Q(x, \partial)e_j = 0, j = 1, \dots, s\}.$$

Put $\mathfrak{A}_k := \mathcal{D}_k \cap \mathfrak{A}$ and consider the associated graded ideal in $\text{gr}(\mathcal{D})$:

$$\text{gr}_{\mathcal{D}}(\mathfrak{A}) = \bigoplus_{k \geq 0} \mathfrak{A}_k / \mathfrak{A}_{k-1}$$

containing the principal symbols of the differential operators in \mathfrak{A} . Clearly, $\text{gr}_{\mathcal{D}}(\mathfrak{A})$ is a homogeneous ideal in the polynomial ring $\mathbb{C}[x, \xi] \cong \text{gr}(\mathcal{D})$, and therefore it can be presented in the form

$$(7) \quad \text{gr}_{\mathcal{D}}(\mathfrak{A}) = \bigoplus_{k \geq 0} I_k(\mathfrak{A}) \xi^k.$$

Here, $I_0(\mathfrak{A}) \subset I_1(\mathfrak{A}) \subset I_2(\mathfrak{A}) \dots$ is an *increasing* sequence of ideals in $\mathbb{C}[x]$ that corresponds to leading terms of the differential operators in \mathfrak{A} . Since $\mathbb{C}[x]$ is a principal ideal domain (recall, $n = 1$), each $I_k(\mathfrak{A})$ is cyclic while the sequence $\{I_k\}$ stabilizes: $I_q = I_{q+1} = I_{q+2} = \dots$ starting from some $q \geq 0$. We write $\tau_k = \tau_k(x)$ for the generators of $I_k(\mathfrak{A})$ in $\mathbb{C}[x]$, and notice that each τ_k is determined uniquely (up to a constant factor). Let

$$p := \min \{k \in \mathbb{Z}_+ : I_k(\mathfrak{A}) \neq 0\}$$

and

$$q := \min \{k \in \mathbb{Z}_+ : I_k = I_l \forall l \geq k\}$$

so that $q \geq p$. Then, the polynomials $\{\tau_p(x)\xi^p, \dots, \tau_q(x)\xi^q\}$ generate the graded ideal $\text{gr}_{\mathcal{D}}(\mathfrak{A})$ in $\mathbb{C}[x, \xi]$. We refer to the integers p and $\alpha_q := \deg \tau_q(x)$ as the *order* and *degree* of the ideal \mathfrak{A} respectively, and write $\text{Ord } \mathfrak{A} := p$, $\text{Deg } \mathfrak{A} := \alpha_q$. A set of differential operators $\{L_p(x, \partial), \dots, L_q(x, \partial)\} \subset \mathfrak{A}$ such that $\sigma(L_r) = \tau_r(x)\xi^r$, $r = p, p+1, \dots, q$, is called a *standard basis* of the ideal \mathfrak{A} (cf. [14]). The differential operator L_p of the minimal order in a standard basis is determined uniquely, once a normalization of its leading coefficient is fixed. It follows from our previous discussion that a standard basis is indeed a system of generators of the ideal \mathfrak{A} . In fact, it is not difficult to prove that $\mathfrak{A} = \mathcal{D}\langle L_p \rangle + \mathcal{D}\langle L_q \rangle$ in agreement with a well-known result of Stafford that every ideal in the Weyl algebra $A_n(\mathbb{C})$ can be generated by two elements (see [12], Ch. I).

3.3. Wilson's Construction. Let $M = \bigoplus_{j=1}^N \Delta(\lambda_j)$ be a direct sum of finite left \mathcal{D} -modules with point support. It is easy to see that $V_M = \bigcup T_{\lambda_j}^* \mathbb{C}$ and, hence, that $\text{supp } M = \{\lambda_1, \dots, \lambda_N\}$. We fix a \mathbb{C} -linear finite-dimensional vector space $\mathcal{V} \subset M$, *homogeneous* with respect to the direct decomposition above: $\mathcal{V} = \bigoplus_j \Delta(\lambda_j) \cap \mathcal{V}$, and such that $M = \mathcal{D}\langle \mathcal{V} \rangle$. Then, according to Lemma 3.1, \mathcal{V} has a finite basis of the form $e_j \cong p_j(\partial) + \mathcal{D}m_{\lambda_j}$. Following Wilson [37], we also consider the subring $\mathcal{R}_{\mathcal{V}} \subset \mathbb{C}[x]$ of polynomials $P \in \mathbb{C}[x]$ with the property $P\mathcal{V} \subset \mathcal{V}$, i.e. $P \in \text{End}_{\mathbb{C}}(\mathcal{V})$. Since $\text{supp } M$ is finite, $\mathcal{R}_{\mathcal{V}}$ is not empty. In particular, it contains all $P \in \bigcap_j \mathfrak{m}_{\lambda_j}^k$ for a sufficiently large $k \in \mathbb{Z}_+$. The left ideal $\mathfrak{A} := \text{Ann}_{\mathcal{D}}(\mathcal{V})$ annihilating \mathcal{V} has the structure of a *right* module over $\mathcal{R}_{\mathcal{V}}$. Indeed, $(\mathfrak{A}.P)\mathcal{V} = \mathfrak{A}(P\mathcal{V}) \subset \mathfrak{A}\mathcal{V} = 0$ so that $\mathfrak{A} \supset \mathfrak{A}.P$ by definition of \mathfrak{A} . Note also that $\text{Ord } \mathfrak{A} = 0$.

Now we apply the operation of a *Fourier transform* $M \rightarrow M^{\vee}$ to the \mathcal{D} -module M . By definition, this is a (unique) bijection induced by the formal involution on the Weyl algebra $\mathcal{D} = A_1(\mathbb{C})$: $x \mapsto -\partial_x$, $\partial_x \mapsto x$. One can prove [10] that $M^{\vee} \in \mathcal{B}_1$ provided $M \in \mathcal{B}_1$. If $\mathcal{V}^{\vee} \subset M^{\vee}$ is the Fourier image of $\mathcal{V} \subset M$, then \mathcal{V}^{\vee} is annihilated by \mathfrak{A}^{\vee} , the image of \mathfrak{A} under the Fourier involution in \mathcal{D} . The fact that $\text{Ord } \mathfrak{A} = 0$ implies that $\mathfrak{A}^{\vee} \cap \mathbb{C}[\partial]$ is not empty. In particular, $\text{Deg } \mathfrak{A}^{\vee} = 0$.

Let $\{G_p(x, \partial), \dots, G_q(x, \partial)\}$ be a standard basis of the ideal \mathfrak{A}^{\vee} so that $\mathfrak{A}^{\vee} = \mathcal{D}\langle G_p \rangle + \mathcal{D}\langle G_q \rangle$. Clearly, we may choose $G_q \in \mathbb{C}[\partial]$, while $G_p(x, \partial) \notin \mathbb{C}[\partial]$ in view of the *homogeneity* of the space \mathcal{V} . Recall that the operator G_p is determined *uniquely* by the ideal \mathfrak{A}^{\vee} (up to an inessential constant factor). We call the leading term of this operator $\tau = \tau_p(x)$ the τ -*function* associated with the ideal \mathfrak{A}^{\vee} . The idea now is to localize the Weyl algebra $A_1(\mathbb{C})$ at $\tau = \tau_p(x)$ and to consider the *extension* of the ideal \mathfrak{A}^{\vee} in this localization.

Thus, we set

$$\mathcal{D}_{\tau} := \mathcal{D}[\tau^{-1}] \cong \mathbb{C}[x, \tau^{-1}] \otimes_{\mathbb{C}[x]} A_1(\mathbb{C})$$

and denote by $\mathfrak{A}_{\tau}^{\vee}$ the left ideal in \mathcal{D}_{τ} generated by \mathfrak{A}^{\vee} . It follows that $\mathfrak{A}_{\tau}^{\vee}$ is a (principal) ideal with a unique generator $D = D(x, \partial) := \tau(x)^{-1} \circ G_p(x, \partial) \in \mathcal{D}_{\tau}$, since $\ker G_p(x, \partial) \subset \ker G_q(\partial)$ by construction. On the other hand, $\mathfrak{A}_{\tau}^{\vee}$ has the structure of a *right* module over the (commutative) ring $\mathcal{R}_{\mathcal{V}}^{\vee} \subset \mathbb{C}[\partial]$, the Fourier transform of $\mathcal{R}_{\mathcal{V}} \subset \mathbb{C}[x]$. Hence, we conclude that for every $L_0 \in \mathcal{R}_{\mathcal{V}}^{\vee}$ there exists an operator $L \in \mathcal{D}_{\tau}$ such that $D \circ L_0 = L \circ D$ in $\mathfrak{A}_{\tau}^{\vee}$. The family of all differential operators $L \in \mathcal{D}_{\tau}$ obtained in this way forms a *commutative* ring in the algebra $\mathcal{D}(X)$ of regular differential operators on $X = \mathbb{C}^1 \setminus \tau^{-1}(0)$, isomorphic to $\mathcal{R}_{\mathcal{V}}$. It is not difficult to show (cf., [37]) that $\text{Spec } \mathcal{R}_{\mathcal{V}}$ is a rational curve with cuspidal singularities located at the points of $\text{supp } M$. In this way, we recover Wilson's scheme of rational Darboux transformations in dimension $n = 1$ (cf. [1, 22]).

4. ALGEBRAIC DARBOUX TRANSFORMATIONS FOR PARTIAL DIFFERENTIAL OPERATORS

The purpose of this section is to generalize Wilson’s construction to higher dimensions. We propose a procedure for generating (super)complete commutative rings of partial differential operators in arbitrary dimension whose common (formal) eigenfunction $\psi = \psi(x, \xi)$ (being necessarily unique) has a particularly simple structure. By analogy with the one-dimensional case, we call this procedure the *algebraic Darboux transformation*. The fact that, in general, $\psi(x, \xi)$ has no trivial symmetry between spatial and spectral variables leads to a new example of the *bispectral duality* (in the sense of [18, 19, 37]) in the case of several variables. More precisely, we extend the algebraic Darboux transformation to its “bispectral” counterpart and construct a commutative ring of (partial) differential operators in the spectral parameter sharing the same eigenfunction $\psi = \psi(x, \xi)$. However, unlike the one dimensional case, we are far from claiming that our procedure completely settles the bispectral problem in many dimensions. Our intention is to explore some first instructive examples in this direction. (For some other examples, see [9].)

4.1. Commuting differential operators associated with \mathcal{D} -modules supported by an algebraic hypersurface.

4.1.1. *The multiplicity free case.* Let $q(x) \in \mathbb{C}[x]$ be an irreducible polynomial in $n \geq 1$ variables, $x := (x_1, \dots, x_n)$, and let \mathfrak{m}_q stand for the corresponding prime ideal in the polynomial ring. Extending the procedure developed in Section 3, we start with the \mathcal{D} -module $\Delta(q) := \mathcal{D}/\mathcal{D}\mathfrak{m}_q$ on \mathbb{C}^n supported at the zero set of q . We single out a cyclic submodule $M := \mathcal{D}\langle e \rangle$ of $\Delta(q)$ generated by the element $e := \tau(-\partial) + \mathcal{D}\mathfrak{m}_q$ where $\tau = \tau(x)$ is an arbitrary fixed polynomial in $\mathbb{C}[x]$, and write $\mathfrak{A} := \text{Ann}_{\mathcal{D}}(e)$ for the left annihilator of e in \mathcal{D} . Clearly, \mathfrak{A} is a left ideal in the Weyl algebra of zero order, $\text{Ord } \mathfrak{A} = 0$, since $\mathfrak{m}_q^N e = 0$ in $\Delta(q)$ for all sufficiently large $N \in \mathbb{Z}_+$ ($N \geq \deg \tau + 1$).

Let $\tilde{\mathcal{R}}_e$ be the subring of differential operators in \mathcal{D} such that

$$\tilde{\mathcal{R}}_e := \{Q \in \mathcal{D} : e \in \ker_M(Q - \lambda) \text{ for some } \lambda \in \mathbb{C}\}.$$

Put $\mathcal{R}_e := \tilde{\mathcal{R}}_e \cap \mathbb{C}[x]$. Then, \mathcal{R}_e is *not empty* since $\mathfrak{A} \subset \tilde{\mathcal{R}}_e$ while $\mathfrak{A} \cap \mathbb{C}[x]$ contains sufficiently large powers of the ideal \mathfrak{m}_q .

As in dimension $n = 1$, we apply to M the formal Fourier transform: $M \rightarrow M^\vee$ and similarly denote by $\mathfrak{A}^\vee, \mathfrak{m}_q^\vee, \mathcal{R}_e^\vee, \dots$ the objects dual to $\mathfrak{A}, \mathfrak{m}_q, \mathcal{R}_e, \dots$ under the corresponding Fourier involution¹. Again, since $\text{Ord } \mathfrak{A} = 0$, it is clear that $\mathfrak{A}^\vee \cap \mathbb{C}[\partial]$ is not empty. Furthermore, \mathfrak{A}^\vee is a *right* module over the (commutative) ring \mathcal{R}_e^\vee .

Let X_τ be the open (quasi)-affine algebraic variety in \mathbb{C}^n obtained by removing the zero set of the polynomial $\tau(x)$, i.e. $X_\tau := \mathbb{C}^n \setminus \tau^{-1}(0)$. Then (see Example 2, Sect. 2.1), we can identify the ring $\mathcal{D}(X_\tau)$ of regular differential operators on X_τ with the localization of the Weyl algebra at $\tau = \tau(x)$, i.e., $\mathcal{D}_\tau := \mathcal{D}(X_\tau) \cong \mathbb{C}[x, \tau^{-1}] \otimes_{\mathbb{C}[x]} A_n(\mathbb{C})$. Since $\mathcal{D} \subset \mathcal{D}_\tau$, one can define the *extension ideal* \mathfrak{A}_τ^\vee of \mathfrak{A}^\vee in this localization, namely $\mathfrak{A}_\tau^\vee := \mathcal{D}_\tau(\mathfrak{A}^\vee)$. The following lemma clarifies the algebraic structure of \mathfrak{A}_τ^\vee .

Lemma 4.1. \mathfrak{A}_τ^\vee is a principal left ideal in \mathcal{D}_τ .

¹Here, $\mathfrak{m}_q, \mathcal{R}_e, \dots$ are regarded as being imbedded in the Weyl algebra $A_n(\mathbb{C})$.

Proof. By construction, \mathfrak{A}^\vee is a left ideal in \mathcal{D} annihilating the principal generator $e^\vee := \tau(x) + \mathcal{D}\mathfrak{m}_q^\vee$ of the \mathcal{D} -module M^\vee . Fix $N \in \mathbb{Z}_+$ such that $N > \deg q$, and define $G(x, \partial) := \tau^N(x) \circ q(\partial) \circ \tau(x)^{-1} \in \mathcal{D}_\tau$. Then, it follows that $G(x, \partial) \in \mathfrak{A}^\vee$. Indeed, in view of the Weyl algebra commutation relations one has $\text{ad}_\tau^N[q(\partial)] = 0$, where ad_τ^N stands for the N -iterated commutator with $\tau = \tau(x)$. Since

$$\text{ad}_{\tau(x)}^N[q(\partial)] = \sum_{k=0}^N (-1)^k \binom{N}{k} \tau(x)^{N-k} \circ q(\partial) \circ \tau(x)^k,$$

the expression $\tau(x)^N \circ q(\partial)$ is divisible by $\tau(x)$ from the right within $A_n(\mathbb{C})$. On the other hand, it is obvious that $G(x, \partial) \circ \tau(x) \equiv 0 \pmod{\mathcal{D}\mathfrak{m}_q^\vee}$.

Put $D = D(x, \partial) := \tau(x)^{-N+1} \circ G(x, \partial)$, so that $D \in \mathfrak{A}_\tau^\vee$ in view of the definition of \mathfrak{A}_τ^\vee . Then, we claim that the ideal \mathfrak{A}_τ^\vee is generated by D in \mathcal{D}_τ . Indeed, if $Q = Q(x, \partial) \in \mathfrak{A}_\tau^\vee$, then Q can be presented as $Q(x, \partial) = \sum_j S_j(x, \partial) \circ T_j(x, \partial)$ with $T_j \in \mathfrak{A}^\vee$ and $S_j \in \mathcal{D}_\tau$. By definition of \mathfrak{A}^\vee , we have $T_j(x, \partial) \circ \tau(x) \equiv 0 \pmod{\mathcal{D}\mathfrak{m}_q^\vee}$ and hence $T_j(x, \partial) \circ \tau(x) = \tilde{T}_j(x, \partial) \circ q(\partial)$ for some $\tilde{T}_j(x, \partial) \in \mathcal{D}$. Multiplying both sides of the last equation by $S_j(x, \partial)$ on the left, by $\tau(x)^{-1}$ on the right and summing up, we get $Q(x, \partial) = \tilde{T}'(x, \partial) \circ D(x, \partial)$ where $\tilde{T}' := \sum_j \tilde{T}_j \circ \tau(x)^{-1} \in \mathcal{D}_\tau$. Thus, \mathfrak{A}_τ^\vee is a cyclic left ideal in \mathcal{D}_τ . \square

Now we are ready to state the main theorem which completes our construction of algebraic Darboux transformations related to the \mathcal{D} -module $\Delta(q)$.

Theorem 4.2. *Associated with $M \subset \Delta(q)$ there exists a commutative ring $\mathcal{R}_M \subset \mathcal{D}_\tau$ of regular differential operators on $X_\tau = \mathbb{C}^n \setminus \tau^{-1}(0)$ isomorphic to \mathcal{R}_e . If $\Lambda(q)$ and $\Lambda(\tau)$ are the complex linealities² of the polynomial q and τ and if $\Lambda(q) \cup \Lambda(\tau) \neq \mathbb{C}^n$, the ring \mathcal{R}_M is nontrivial in the sense that \mathcal{R}_M cannot be transformed into $\mathcal{R}_e^\vee \subset \mathbb{C}[\partial]$ by conjugation by a regular function on X_τ or through a local nonsingular change of variables.*

Proof. By definition, \mathfrak{A}_τ^\vee is generated by \mathfrak{A}^\vee as a left module over \mathcal{D}_τ . Hence, it inherits (from \mathfrak{A}^\vee) the structure of a right module over the commutative ring \mathcal{R}_e^\vee . Therefore, in view of Lemma 4.1, we conclude that for every $L_0 \in \mathcal{R}_e^\vee$ there exists an operator $L \in \mathcal{D}_\tau$ such that $D \circ L_0 = L \circ D$ in \mathfrak{A}_τ^\vee with $D = \tau(x) \circ q(\partial) \circ \tau(x)^{-1}$. Denote by \mathcal{R}_M the set of all differential operators $L \in \mathcal{D}_\tau$ constructed through this procedure. Then, \mathcal{R}_M forms a commutative subalgebra in the ring $\mathcal{D}(X_\tau)$ since the latter has no zero divisors. When $\Lambda(q) \cup \Lambda(\tau) \neq \mathbb{C}^n$, $\Lambda^\perp(q) \cap \Lambda^\perp(\tau)$ is not empty and $[q(\partial), \tau(x)] \neq 0$, so that $D \notin \mathbb{C}[\partial]$. This implies that \mathcal{R}_M is neither equal nor conjugate to \mathcal{R}_e^\vee . Moreover, since the operators in \mathcal{R}_M have constant principal symbols, they cannot be reduced to their principal parts through a local change of coordinates. \square

Remark 4.3. Recall that, by construction, \mathcal{R}_e (and hence \mathcal{R}_e^\vee) is not empty. In particular, \mathcal{R}_e contains the powers $\mathfrak{m}_q^N \supset \mathfrak{m}_q^{N+1} \supset \dots$ of the ideal \mathfrak{m}_q starting with $N = \deg \tau(x) + 1$. By Fourier duality, this implies that $(\mathfrak{m}_q^\vee)^N \subseteq \mathcal{R}_e^\vee$ for all $N \geq \deg \tau(x) + 1$. In fact, if $L_0 \in (\mathfrak{m}_q^\vee)^N$, then the corresponding operator $L \in \mathcal{R}_M$ can be constructed explicitly. Indeed, we have $L_0 = p(\partial) \circ q(\partial)^N$ with $p(\partial) \in \mathbb{C}[\partial]$.

²By a complex lineality of a polynomial $P \in \mathbb{C}[x]$ we mean the maximal linear subspace $\Lambda(P)$ of \mathbb{C}^n such that P can be restricted to a polynomial on the quotient $\mathbb{C}^n / \Lambda(P)$.

On the other hand, in view of the Weyl algebra commutation relations, the identity

$$\text{ad}_{q(\partial)}^N[\tau(x)] := [q(\partial), [q(\partial), \dots [q(\partial), \tau(x)] \dots]] = 0,$$

or more explicitly,

$$\sum_{k=0}^N (-1)^k \binom{N}{k} q(\partial)^{N-k} \circ \tau(x) \circ q(\partial)^k = 0$$

holds for all $N \geq \deg \tau + 1$. Whence, we get

$$q(\partial)^N \circ \tau(x) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} q(\partial)^{N-k} \circ \tau \circ q(\partial)^k,$$

so that L_0 can be factorized in the form

$$(8) \quad L_0 = Q(x, \partial) \circ D(x, \partial).$$

Here

$$(9) \quad Q(x, \partial) := \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} p(\partial) q(\partial)^{N-k} \circ \tau(x) \circ q(\partial)^{k-1} \circ \tau(x)^{-1}$$

and $D(x, \partial) := \tau(x) \circ q(\partial) \circ \tau(x)^{-1}$. By interchanging the factors in (8), we obtain the corresponding operator in \mathcal{R}_M :

$$(10) \quad L := D(x, \partial) \circ Q(x, \partial).$$

This procedure is a *precise* analog of the classical one-dimensional Darboux transformation [17].

4.1.2. The general case. This procedure of algebraic Darboux transformations can be extended to the case of arbitrary multiplicities. Again, we start with a pair of complex polynomials $q = q(x)$ and $\tau = \tau(x)$ in $n \geq 1$ independent variables (without assuming q to be primitive). In this case, we can decompose $q = C q_1^{k_1} q_2^{k_2} \dots q_s^{k_s}$ into a product of powers of pairwise distinct irreducible polynomials $q_i = q_i(x) \in \mathbb{C}[x]$ with (strictly) positive integer multiplicities $k_i \in \mathbb{Z}_+$, $C \in \mathbb{C}^\times$, and we write $\mathfrak{m}_i := \mathfrak{m}_{q_i}$ for the corresponding prime ideals in $\mathbb{C}[x]$. To each k_i we then associate the \mathbb{C} -linear vector space $\mathcal{V}_i \subset \mathbb{C}[x]$ spanned by all multiples of $\tau(x)$ by polynomials of order less than k_i , i.e. we set

$$\mathcal{V}_i = \text{span}_{\mathbb{C}} \langle \alpha(x) \tau(x) \in \mathbb{C}[x] : \deg \alpha < k_i \rangle.$$

With the notation of Section 4.1.1 we define $\Delta(q_1, \dots, q_s) := \bigoplus_{i=1}^s \Delta(q_i)$ as a left \mathcal{D} -module over the Weyl algebra on \mathbb{C}^n and consider its submodule M generated by the elements $e_i := p_i(-\partial) + \mathcal{D}\mathfrak{m}_i$ with all $p_i \in \mathcal{V}_i$, $i = 1, \dots, s$.

Let M^\vee be a Fourier transform of the \mathcal{D} -module M . Clearly, $M^\vee = \mathcal{D}\langle e_1^\vee, \dots, e_s^\vee \rangle$, where $e_i^\vee := \mathcal{V}_i + \mathcal{D}\mathfrak{m}_i^\vee$, while \mathfrak{m}_i^\vee stands for the image of \mathfrak{m}_i under the (formal) Fourier involution in \mathcal{D} . We write $\mathfrak{A}^\vee \subset \mathcal{D}$ for the left ideal in \mathcal{D} annihilating the elements e_i^\vee , $i = 1, \dots, s$, $\mathfrak{A}^\vee := \text{Ann}_{\mathcal{D}}(e_1^\vee, \dots, e_s^\vee)$ and consider its extension $\mathfrak{A}_\tau^\vee := \mathcal{D}_\tau \langle \mathfrak{A}^\vee \rangle$ in the localization of the Weyl algebra \mathcal{D}_τ at τ .

In this setting, we can refine the argument leading to the proof of Lemma 4.1, i.e. we can show the *cyclicity* of the ideal \mathfrak{A}_τ^\vee within \mathcal{D}_τ . Indeed, if $Q(x, \partial) \in \mathfrak{A}_\tau^\vee$, one has $Q(x, \partial) = \sum S(x, \partial) \circ T(x, \partial)$ with $S(x, \partial) \in \mathcal{D}_\tau$ and $T(x, \partial) \in \mathfrak{A}^\vee \subset \mathcal{D}$. The latter implies that $T(x, \partial) \circ p(x) = \tilde{T}_p(x, \partial) \circ q_i(\partial)$ for *any* $p(x) \in \mathcal{V}_i$ and some $\tilde{T}_p \in \mathcal{D}$. Since $p(x) = \alpha(x) \tau(x)$ with $\alpha(x) \in \mathbb{C}[x]$, $\deg \alpha < k_i$, we can

rewrite the above equation in the form $R(x, \partial) \circ \alpha(x) = \tilde{T}_p(x, \partial) \circ q_i(\partial)$, where $R(x, \partial) := T(x, \partial) \circ \tau(x) \in \mathcal{D}_\tau$. It follows that all the derivatives of the *full* symbol $R(x, \xi)$ of the operator R up to order $k_i - 1$ with respect to ξ vanish on the zero set of the ideal \mathfrak{m}_i in \mathbb{C}^n . Elementary induction in k_i with the use of the Nullstellensatz gives immediately that $R(x, \xi) = R_i(x, \xi)q_i^{k_i}(\xi)$ and, hence, $T(x, \partial)$ is divisible on the right by $q_i(\partial)^{k_i} \circ \tau(x)^{-1}$ for all $i = 1, 2, \dots, s$. Finally, due to the commutativity of the operators $D_i := \tau(x) \circ q_i(\partial)^{k_i} \circ \tau(x)^{-1} \in \mathcal{D}_\tau$, we conclude that T must be of the form $T(x, \partial) = \tilde{T}'(x, \partial) \circ D(x, \partial)$ where $D := \prod_i D_i = \tau(x) \circ q(\partial) \circ \tau(x)^{-1}$ and $\tilde{T}' \in \mathcal{D}_\tau$. In this way, the operator $D(x, \partial)$ is a principal generator of the ideal \mathfrak{A}_τ^\vee in \mathcal{D}_τ .

Let $\mathcal{R}_e := \{P \in \mathbb{C}[x] : P \cdot \text{span}_{\mathbb{C}}(e_1, \dots, e_s) \subseteq \text{span}_{\mathbb{C}}(e_1, \dots, e_s)\}$ and let \mathcal{R}_e^\vee be its formal Fourier transform. Both \mathcal{R}_e and \mathcal{R}_e^\vee are not empty: they contain in particular the ideals $\cap \mathfrak{m}_i^{k_i}$ and $\cap (\mathfrak{m}_i^\vee)^{k_i}$ respectively. Then, the left ideals \mathfrak{A}^\vee and \mathfrak{A}_τ^\vee have the structure of right modules over the commutative ring \mathcal{R}_e^\vee , and therefore we again arrive at the result stated in Theorem 4.2. Namely, we obtain a commutative ring $\mathcal{R}_M \subset \mathcal{D}_\tau$ of regular differential operators on $X_\tau = \mathbb{C}^n \setminus \tau^{-1}(0)$ isomorphic to \mathcal{R}_e^\vee . This completes the procedure of algebraic Darboux transformations in the case of arbitrary multiplicities.

4.2. The Bispectral Duality. The formal spectral problem for the commutative ring \mathcal{R}_M has a unique solution of a particularly simple structure. Indeed, by construction, the (quasi-)regular function on T^*X_τ given by $\psi = \psi(x, \xi) := D(x, \partial)e^{(x, \xi)}$ is a common eigenfunction for all differential operators in \mathcal{R}_M . We call ψ the *Baker-Akhiezer function* associated with \mathcal{R}_M .

The following elementary argument shows that there exists a certain complete commutative ring \mathcal{R}_M^b of partial differential operators on $X_q = \mathbb{C}^n \setminus q^{-1}(0)$ written in terms of the spectral variable ξ for which ψ is also a (unique) common eigenfunction.

Put $N = \deg q(x)$ and denote by I^b the principal ideal in the polynomial ring $\mathbb{C}[x]$ generated by $\tau(x)^N$. For every $\gamma \in I^b$ we have $\gamma(x)\psi(x, \xi) = Q_\gamma(x, \partial_x)e^{(x, \xi)}$ with some $Q_\gamma \in A_n(\mathbb{C})$. Then, there obviously exists another differential operator $Q_\gamma^b(\xi, \partial_\xi) \in A_n(\mathbb{C})$ written formally in terms of ξ such that $Q_\gamma(x, \partial_x)e^{(x, \xi)} = Q_\gamma^b(\xi, \partial_\xi)e^{(x, \xi)}$, and hence

$$(11) \quad Q_\gamma^b(\xi, \partial_\xi)e^{(x, \xi)} = \gamma(x)\psi(x, \xi).$$

On the other hand, we have

$$(12) \quad \tau(\partial_\xi)\psi(x, \xi) = \tau(x)q(\partial_x)e^{(x, \xi)} = q(\xi)\tau(x)e^{(x, \xi)}$$

in view of the definition of ψ . It follows immediately from (11) and (12) that

$$(13) \quad L_\gamma(\xi, \partial_\xi)\psi(x, \xi) = \gamma(x)\tau(x)\psi(x, \xi),$$

where $L_\gamma(\xi, \partial_\xi) := Q_\gamma^b(\xi, \partial_\xi) \circ q(\xi)^{-1} \circ \tau(\partial_\xi)$. The family of all operators $L_\gamma(\xi, \partial_\xi)$ constructed in this way forms a commutative ring \mathcal{R}_M^b of partial differential operators in the localized Weyl algebra $\mathcal{D}_q \cong \mathbb{C}[\xi, q(\xi)^{-1}] \otimes_{\mathbb{C}[\xi]} A_n(\mathbb{C})$ isomorphic to I^b . This ring can be regarded as a *bispectral dual* of \mathcal{R}_M in the sense of [18, 37]. The simple procedure we have applied to generate the ring \mathcal{R}_M^b is, in fact, an extension of the bispectral Darboux transformation scheme in $n = 1$ as discussed in [22, 24].

In conclusion, we note that the ring \mathcal{R}_M^b , though complete, is not necessarily a *maximal commutative* subring in \mathcal{D}_q , i.e., the centralizer $Z(\mathcal{R}_M^b)$ of \mathcal{R}_M^b may

happen to be larger than \mathcal{R}_M^b itself. It is an interesting problem to clarify the structure of $Z(\mathcal{R}_M^b)$ in general.

5. EXAMPLES, APPLICATIONS AND CONCLUDING REMARKS

The purpose of this section is to display some explicit examples of the theory presented so far and to illuminate some of its implications.

Let $n = 2$. We choose $\tau(x_1, x_2) = x_1^2 - x_2$ and $q(x_1, x_2) = x_1x_2 - \lambda$, $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, the polynomial q is irreducible. According to Theorem 4.2, for any constant coefficient operator $Q \in \mathbb{C}[\partial_1, \partial_2]$ and $L_0 = (\partial_1\partial_2 - \lambda)^3$ there exists a unique partial differential operator L_Q satisfying the identity

$$(14) \quad D \circ Q \circ L_0 = L_Q \circ D,$$

where

$$(15) \quad D := \tau(x_1, x_2) \circ q(\partial_1, \partial_2) \circ \tau(x_1, x_2)^{-1} \\ = \left(\partial_1 - \frac{2x_1}{x_1^2 - x_2} \right) \left(\partial_2 + \frac{1}{x_1^2 - x_2} \right) - \lambda.$$

For example, if $Q \equiv 1$, formulas (8)–(10) (see Remark 4.3) give explicitly

$$(16) \quad L_1 = \partial_1^3 \partial_2^3 + \frac{-6(x_1^2 + x_2)}{(x_1^2 - x_2)^2} \partial_1 \partial_2^3 + \frac{24x_1x_2}{(x_1^2 - x_2)^3} \partial_2^3 \\ + \frac{-3(-4x_1 + \lambda x_1^4 - 2\lambda x_1^2 x_2 + \lambda x_2^2)}{(x_1^2 - x_2)^2} \partial_1^2 \partial_2^2 + \frac{-6(13x_1^2 + 5x_2)}{(x_1^2 - x_2)^3} \partial_1 \partial_2^2 \\ + \frac{6(18x_1^3 + \lambda x_1^6 + 30x_1x_2 - \lambda x_1^4 x_2 - \lambda x_1^2 x_2^2 + \lambda x_2^3)}{(x_1^2 - x_2)^4} \partial_2^2 + \frac{-3}{(x_1^2 - x_2)^2} \partial_1^3 \partial_2 \\ + \frac{54x_1}{(x_1^2 - x_2)^3} \partial_1^2 \partial_2 + \frac{3(6\lambda^2 x_1^4 x_2^2 - 4\lambda^2 x_1^2 x_2^3 + \lambda^2 x_2^4)}{(x_1^2 - x_2)^4} \partial_1 \partial_2 \\ + \frac{3(-118x_1^2 - 4\lambda x_1^5 + \lambda^2 x_1^8 - 26x_2 + 8\lambda x_1^3 x_2 - 4\lambda^2 x_1^6 x_2 - 4\lambda x_1 x_2^2)}{(x_1^2 - x_2)^4} \partial_1 \partial_2 \\ + \frac{6(136x_1^3 + 9\lambda x_1^6 + 104x_1x_2 - 15\lambda x_1^4 x_2 + 3\lambda x_1^2 x_2^2 + 3\lambda x_2^3)}{(x_1^2 - x_2)^5} \partial_2 + \frac{-3}{(x_1^2 - x_2)^3} \partial_1^3 \partial_2 \\ + \frac{3(24x_1 + \lambda x_1^4 - 2\lambda x_1^2 x_2 + \lambda x_2^2)}{(x_1^2 - x_2)^4} \partial_1^2 \\ + \frac{-12(52x_1^2 + 3\lambda x_1^5 + 8x_2 - 6\lambda x_1^3 x_2 + 3\lambda x_1 x_2^2)}{(x_1^2 - x_2)^5} \partial_1 \\ + \frac{1920x_1^3 - \lambda^3 x_1^{12} + 960x_1x_2 + 6\lambda^3 x_1^{10} x_2 - 15\lambda^3 x_1^8 x_2^2 + 24\lambda x_2^3}{(x_1^2 - x_2)^6} \\ + \frac{-\lambda^3 x_2^6 + 20\lambda x_1^6 (6 + \lambda^2 x_2^3) + 6\lambda x_1^2 x_2^2 (12 + \lambda^2 x_2^3) - 3\lambda x_1^4 x_2 (72 + 5\lambda^2 x_2^3)}{(x_1^2 - x_2)^6}.$$

Note that the operators L_Q are nontrivial in the sense that they cannot be reduced to the constant operators $L_Q^0 := Q \circ L_0$ by means of conjugation or change of variables.

The simplest operator in the spectral parameter which has the same eigenfunction as the L_Q 's can also be computed in an explicit form using (11), (12) and (13). Namely, we have

$$(17) \quad L_1^b(\xi, \partial_\xi) = Q_1^b \circ q(\xi_1, \xi_2)^{-1} \circ \tau(\partial_\xi),$$

where

$$\begin{aligned} Q_1^b(\xi, \partial_\xi) &= (\xi_1 \xi_2 - \lambda) \partial_{\xi_1}^4 - 2\xi_2 \partial_{\xi_1}^3 - 2(\xi_1 \xi_2 - \lambda) \partial_{\xi_1}^2 \partial_{\xi_2} \\ &\quad + \xi_1 \partial_{\xi_1}^2 + (\xi_1 \xi_2 - \lambda) \partial_{\xi_2}^2 + 2\xi_2 \partial_{\xi_1} \partial_{\xi_2} - \xi_1 \partial_{\xi_2} - 4\partial_{\xi_1}. \end{aligned}$$

The next example reveals the link to the theory of Huygens' principle on Minkowski spaces. Let n be an arbitrary even number, $n \geq 4$. Take $\tau(x) = x_1^2 - x_2$ and $q(x) = x_1 x_2 - x_3^2 - x_4^2 - \dots - x_n^2$. Clearly, q is again an irreducible polynomial and we can apply Theorem 4.2. The simplest operator in the commutative ring \mathcal{R}_M has the same form as (16), where the parameter λ is formally replaced by the Laplacian $\Delta_{n-2} = \partial_3^2 + \partial_4^2 + \dots + \partial_n^2$ in $n-2$ extra variables. This makes sense since λ enters polynomially in to L_0 and L_1 . The operator $\tilde{L}_0 = (\partial_1 \partial_2 - \partial_3^2 - \dots - \partial_n^2)^3$ is a cube of the classical wave operator, and hence, it satisfies Huygens' principle for all *even* $n \geq 8$ (cf., e.g., [31]).

Adapting the technique developed in [7], we can prove that the operator $\tilde{L}_1 = \partial_1^3 \partial_2^3 + \dots$ obtained by the formal substitution $\lambda \mapsto \Delta_{n-2}$ in (16) is also a nontrivial Huygensian hyperbolic operator in even dimensions $n \geq 12$. Note that \tilde{L}_1 *cannot* be presented as a power of the second order wave-type operator on the Minkowski space \mathbb{M}^n . Therefore, we get a new interesting example related to the classical Hadamard's problem (see [6, 20] and references therein).

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